Asian Journal of Statistical Sciences
Vol. 4, No-2, (2024), pp. 63-81

ARF INDIA
Academic Open Access Publishing
Journal homepage link:
http://arfjournals.com/ajss

Parameter Estimation for Parallel-Connected Systems with Uneven Load-Sharing for Censored Data

Santosh S. Sutar[a], Sukumar V. Rajguru[b] and Prajakta S. Patil[c]

[a,c] *Yashwantrao Chavan School of Rural Development, Shivaji University, Kolhapur - 416 004, India.* [b]*Department of Statistics, Shivaji University, Kolhapur - 416 004, India.*

ARTICLE HISTORY
Compiled April 2, 2025

Received 05 June 2024; Accepted 21 September 2024

ABSTRACT
We consider a parallel-connected configuration comprised of multiple k-out-of-m load sharing subsystems, employing proportional conditional failure rates and the unequal load-sharing principle to model component failures. Our study focuses on estimating load sharing parameters using censored Type-I and Type-II failure data, considering both Exponential and Weibull baseline distributions. We examine two scenarios: n parallel subsystems with a 1-out-of-2 setup and with a 2-out-of-4 setup. The accuracy of parameter estimation methods is evaluated through simulation studies and validated using a real-world dataset. This approach is adaptable to configurations with varied load sharing setups, including those in series or a p-out-of-q steup

KEYWORDS
k-out-of-m system, Load sharing system, Order statistics, Proportional conditional failure rate, Reliability modeling, Type-I and Type-II censoring.

1. Introduction

Numerous academic papers using k-of-m solutions for load sharing have been published in the literature. Liu [1], for instance, looked into the dependability of a k-of-n system with a variety of components. A generalized accelerated failure-time model was used in the study to evaluate system performance, taking into account arbitrary distributions for component failure times. In a similar vein, Kim and Kvam [2] examined a multicomponent load-sharing system's performance using a maximum likelihood technique. Using an asymptotic limiting process, Kyam and Pena [3] determined the load sharing parameter values for parallel systems with m identical components.

Singh et al. [4] developed standard errors, hazard rate functions, and system reliability measures while concentrating on Bayesian estimation in parallel systems with k components using load sharing. Tian et al. [5] used a recursive approach to assess the reliability of k-of-n systems with different performance levels, taking into account components that are evenly and independently distributed. A thorough closed-form statement for the lifetime dependability of load-sharing k-out-of-n hybrid redundant

Santosh S. Sutar[a]. Email:sss.ycsrd@unishivaji.ac.in

systems was presented by Huang and Xu [6]. These systems make use of m components that are initially configured as active units. Depending on how a task is carried out, these components switch between processing and wait stages, with each state having an arbitrary failure distribution. They also arrived at a formula for lifetime reliability for these hybrid redundant systems.

In order to model component lifetimes, Singh and Gupta [7] used the Lindley distribution. They also created a Bayesian estimation technique for parameter estimation in a k-out-of-n load-sharing parallel component system. Park [8] examined how well a parallel multiple component load-sharing system performed, determining a number of parameters based on component lifetime distributions by maximum likelihood estimation. In order to develop the best rules for achieving maximum dependability across missions, Hus and Elsayed [9] investigated the implementation of the k-out-of-n system in the aircraft industry.

Kong and Ye [10] developed two interval procedures for load-sharing systems based on failure data, assuming exponentially distributed component lifetimes and employing a log-linear function to model the relationship between lifetimes and stress levels. Xu et al. [11] utilized weighted principal component analysis and a flowgraph combined with multi-response optimization to optimize maintenance module design and repair policies for a load-sharing repairable k-out-of-n system.

Kim [12] focused on determining the optimal value of n to maximize system reliability, considering availability or reliability as performance metrics. Pundir and Gupta[13] estimated the reliability of load-sharing systems with multiple components using real-life data and assuming a Chen distribution for component lifetimes. Neama and Temraz (2018) investigated the performance of a load-sharing system composed of two degraded dependent parallel components. Wang et al. [15] calculated the reliability of memory-based load-sharing models using a nonhomogeneous Poisson process for stochastic external load occurrences.

Sutar and Naik-Nimbalkar [16–18] proposed different load-sharing models based on accelerated failure time, conditional proportional hazard and conditional proportional reverse hazard with various baseline distributions. Ali et al. [20] examined a 1-out-of-k parallel load-sharing model using Bayesian estimation and simulation to compare performance, alongside employing maximum likelihood and Bayesian estimators to solve the models. Sutar et al. [19] utilized an Accelerated Failure Time (AFT)-based load sharing model for a parallel system comprising two components, employing a modified Weibull distribution as the baseline. Recently Pesch et al. [21, 22] have developed a load sharing model through the concept of sequential order statistics (SOSs) or extended SOSs and conditional proportional hazard (CPH) model by assuming that system is composed by heterogeneous components.

The discussed study primarily addresses the analysis of load-sharing system models with complete system failure data, assuming that all systems have failed by the experiment's end. However, in today's context, where companies produce highly reliable systems with extended time to failure, it becomes impractical to draw conclusions based on complete system failure data. In light of this, the present study emphasizes the estimation of load-sharing system model parameters using Type-I and Type-II failure censored data. The assumption here is that the system components exhibit time-varying Exponential and Weibull failure rates. Type-I censoring allows control over the testing time, as the experiment's terminal time is predetermined. On the other hand, Type-II censoring aids in minimizing experimentation costs in terms of failed units, as the test concludes after a predetermined number of failures.

The statistical assumptions and model description is discussed in Section 2. In Section 3, we report the system reliability of a whole system which comprises several k-out-of-m load sharing subsystems arranged in parallel. The parameter estimation procedures for whole system in general setup as well as in particular setup with Exponential and Weibull baseline distributions are discussed in section 4. A simulation study is presented in Section 5. In Section 6, we give an illustration. The conclusions are given in the last section.

2. Statistical Assumptions and Model Description

A system with m components is said to be k-out-of-m if $(m-k+1)$ components are functioning and $(m-k+1)^{\text{th}}$ failure represents the system failure. It is obvious that we have a parallel system for $k=1$ and a series system for $k=m$. Here onward, we refer the entire configuration as 'system' and individual k-out-of-m load-sharing system as a 'subsystem'.

The model is based on the following assumptions.

(i) We set up n parallel k-out-of-m load sharing subsystems that are independently and identically distributed (iid).

(ii) The failure times of the j^{th} component of the i^{th} subsystem is denoted by $T_{(ij)}, i = 1,2,3,\ldots,n; j = 1,2,\ldots,(m-k+1)$.

(iii) The baseline distribution, which is the common (identical) lifetime distribution assumed by all the components of the i^{th} subsystem, is used to describe how independently each component operates.

(iv) The baseline probability density function (pdf), survival function (sf) and failure rate function of m independently working components are denoted by $f(\cdot,\underline{\theta})$, $\bar{F}(\cdot,\underline{\theta})$ and $h(\cdot,\underline{\theta})$ respectively. The parameter $\underline{\theta}$ may be scalar or vector of baseline parameters.

(v) In case of i^{th} subsystem $(i = 1,2,\ldots,n)$,

  (a) $T_{(i1)}$ is the first failure time and its pdf is given by,

$$g(t_{(i1)},\underline{\theta}) = m\left[\bar{F}(t_{(i1)})\right]^{m-1} f(t_{(i1)}), t_{(i1)} \geq 0, j = 1,2,\ldots,(m-k+1).$$

  (b) The conditional failure rate of $(j+1)^{th}$ failure given that j^{th} failure occur at $t_{(ij)}$ is

$$h_{T_{(ij+1)}|T_{(ij)}=t_{(ij)}}(t_{(ij+1)},\underline{\theta}) = (m-j)\beta_j h(t), j = 1,2,\ldots,(m-k), t_{(ij)} \leq t_{(ij+1)}, \beta_j > 0. \quad (1)$$

(vi) The estimation of model parameters is conducted under the presence of Type-I and Type-II censored failure data.

Remark (1): When a component fails, the load increases (i.e., $\beta_j > 1$) or decreases (i.e., $0 < \beta_j < 1$) or the failure rate of the component does not change for the remaining components (i.e., $\beta_j = 1$).

Remark (2) This approach can also be applied to subsystems with different load sharing arrangements, such as those in series or adhering to a p-out-of-q setup.

3. Parameter estimation

The likelihood function corresponding to i^{th} k-out-of-m load sharing subsystem is,

$$L_i = \frac{m!}{(k-1)!} f(t_{(i1)}) \left(\bar{F}(t_{(i1)})\right)^{m-1} \prod_{j=1}^{(m-k)} \left\{ \beta_j \left(\frac{\bar{F}(t_{(ij+1)}, \underline{\theta})}{\bar{F}(t_{(ij)}, \underline{\theta})} \right)^{(m-j)\beta_j - 1} \frac{f(t_{(ij+1)}, \underline{\theta})}{\bar{F}(t_{(ij)}, \underline{\theta})} \right\}, \quad i = 1, 2, \ldots, n. \tag{2}$$

Let $\underline{T} = (T_{i1}, T_{i2}, \ldots, T_{i(m-k+1)};\quad i = 1, 2, \ldots, n)$ be a random sample from 'n' parallelly connected k-out-of-m load sharing subsystems. Then the likelihood under Type-I and Type-II censoring is given below.

3.1. Likelihood function under Type-I Censoring:

Under Type-I censoring the termination time of the experiment has been fixed in advance. Let t_0 be the termination time for the experiment and $p(< n)$ be the subsystems which failure times are observed before t_0. Now, the likelihood function (L_1) when only $p(< n)$ subsystems are uncensored and $(n-p)$ subsystems are censored, is given by,
L_1={Failure rate of p uncensored subsystems}*{survival time of $(n-p)$ censored systems},
$L_1 = \prod_{i=1}^{p} L_i \left\{ \prod_{j=1}^{(m-k)} \bar{G}_{T_{(0j+1)}|T_{(pj)}=t_{(pj)}}(t_{(0j+1)}, \underline{\theta}) \right\}^{n-p}$,

$$L_1 = \left(\frac{m!}{(k-1)!}\right)^p \prod_{i=1}^{p} \left[f(t_{(i1)}) \left(\bar{F}(t_{(i1)})\right)^{m-1} \right] \prod_{i=1}^{p} \left\{ \prod_{j=1}^{(m-k)} \left[\beta_j \left(\frac{\bar{F}(t_{(ij+1)}, \underline{\theta})}{\bar{F}(t_{(ij)}, \underline{\theta})} \right)^{(m-j)\beta_j - 1} \frac{f(t_{(ij+1)}, \underline{\theta})}{\bar{F}(t_{(ij)}, \underline{\theta})} \right] \right\}$$
$$\left[\prod_{j=1}^{(m-k)} \left(\frac{\bar{F}(t_{(0j+1)}, \underline{\theta})}{\bar{F}(t_{(pj)}, \underline{\theta})} \right)^{(m-j)\beta_j} \right]^{(n-p)}.$$

3.2. Likelihood function under Type-II Censoring:

Under Type-II censoring number of failures are decided in advance. The experiment was terminates after a predetermined number of failures are observed. Let $q(< n)$ be the predetermined number of failure subsystems. Thus, the likelihood function (L_2) with q subsystem failure times and $(n-q)$ subsystem censored times is given by,
L_2={Failure rate of q uncensored subsystems}*{Survival time of $(n-q)$ censored systems},
$L_2 = \prod_{i=1}^{q} L_i \left\{ \prod_{j=1}^{(m-k)} \bar{G}_{T_{(qj+1)}|T_{(qj)}=t_{(qj)}}(t_{(qj+1)}, \underline{\theta}) \right\}^{n-q}$,

$$L_2 = \left(\frac{m!}{(k-1)!}\right)^q \prod_{i=1}^{q} \left[f(t_{(i1)}) \left(\bar{F}(t_{(i1)})\right)^{m-1} \right] \prod_{i=1}^{q} \left\{ \prod_{j=1}^{(m-k)} \left[\beta_j \left(\frac{\bar{F}(t_{(ij+1)}, \underline{\theta})}{\bar{F}(t_{(ij)}, \underline{\theta})} \right)^{(m-j)\beta_j - 1} \frac{f(t_{(ij+1)}, \underline{\theta})}{\bar{F}(t_{(ij)}, \underline{\theta})} \right] \right\}$$
$$\left[\prod_{j=1}^{(m-k)} \left(\frac{\bar{F}(t_{(qj+1)}, \underline{\theta})}{\bar{F}(t_{(qj)}, \underline{\theta})} \right)^{(m-j)\beta_j} \right]^{(n-q)}.$$

We can combine both the cases and can write the combined likelihood function as below.

3.3. Combined likelihood function:

The combined likelihood function (L_c) based on likelihood functions of Type-I and Type-II censored data given in (3) and (4) can be given as,

$$L_c = \left(\frac{m!}{(k-1)!}\right)^s \prod_{i=1}^{s}\left[f(t_{(i1)})\left(\bar{F}(t_{(i1)})\right)^{m-1}\right]\prod_{i=1}^{s}\left\{\prod_{j=1}^{(m-k)}\left[\beta_j\left(\frac{\bar{F}(t_{(ij+1)},\underline{\theta})}{\bar{F}(t_{(ij)},\underline{\theta})}\right)^{(m-j)\beta_j-1}\frac{f(t_{(ij+1)},\underline{\theta})}{\bar{F}(t_{(ij)},\underline{\theta})}\right]\right\}$$
$$\left[\prod_{j=1}^{(m-k)}\left(\frac{\bar{F}(C,\underline{\theta})}{\bar{F}(t_{(sj),\underline{\theta}})}\right)^{(m-j)\beta_j}\right]^{(n-s)},$$

where,

$$C = \begin{cases} t_{(0j+1)}; & \text{Type I censoring (time censored)},\\ t_{(qj+1)}; & \text{Type II censoring (failure censored)}, \end{cases}$$

and

$$s = \begin{cases} p; & \text{Type I censoring (time censored)},\\ q; & \text{Type II censoring (failure censored)}. \end{cases}$$

Based on these likelihood, we can estimate the unknown model parameters by using maximum likelihood estimation procedure.

3.4. Maximum Likelihood Estimation

The combined likelihood function based on Type-I and Type-II censoring is,

$$L_c = \left(\frac{m!}{(k-1)!}\right)^s \prod_{i=1}^{s}\left[f(t_{(i1)})\left(\bar{F}(t_{(i1)})\right)^{m-1}\right]\prod_{i=1}^{s}\left\{\prod_{j=1}^{(m-k)}\left[\beta_j\left(\frac{\bar{F}(t_{(ij+1)},\underline{\theta})}{\bar{F}(t_{(ij)},\underline{\theta})}\right)^{(m-j)\beta_j-1}\frac{f(t_{(ij+1)},\underline{\theta})}{\bar{F}(t_{(ij)},\underline{\theta})}\right]\right\}$$
$$\left[\prod_{j=1}^{(m-k)}\left(\frac{\bar{F}(C,\underline{\theta})}{\bar{F}(t_{(sj),\underline{\theta}})}\right)^{(m-j)\beta_j}\right]^{(n-s)}.$$

The log likelihood function is,

$$\begin{aligned}\log L_c &= s\log\left(\frac{m!}{(k-1)!}\right) + \sum_{i=1}^{s}\log f(t_{(i1)}) + (m-1)\sum_{i=1}^{s}\log\bar{F}(t_{(i1)})\\ &+\sum_{i=1}^{s}\left\{\sum_{j=1}^{(m-k)}\left\{\log\beta_j + \left[((m-j)\beta_j-1)\left(\log\bar{F}(t_{(ij+1)},\underline{\theta}) - \log\bar{F}(t_{(ij)},\underline{\theta})\right)\right] + \log f(t_{(ij+1)},\underline{\theta}) - \log\bar{F}(t_{(ij)},\underline{\theta})\right\}\right\}\\ &+\sum_{j=1}^{(m-k)}\left\{(n-s)(m-j)\beta_j\left[\log\bar{F}(C,\underline{\theta}) - \log\bar{F}(t_{(sj)},\underline{\theta})\right]\right\}.\end{aligned}$$

For obtaining maximum likelihood estimators of $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, we have to solve following log-likelihood equations,

$$\frac{\partial\log L_c}{\partial\theta_l} = 0, \quad l = 1, 2, 3, \ldots, k.$$

Similarly, for obtaining maximum likelihood estimator of β_j, we solve following log-likelihood equation,

$$\frac{\partial \log L_c}{\partial \beta_j} = \frac{s}{\beta_j} + (m-j)\sum_{i=1}^{s}\left[\log \bar{F}(t_{(ij+1)},\underline{\theta}) - \log \bar{F}(t_{(ij)},\underline{\theta})\right] + \left\{(n-s)(m-j)\left[\log \bar{F}(C,\underline{\theta}) - \log \bar{F}(t_{(sj)},\underline{\theta})\right]\right\} = 0, \tag{3}$$

which gives,

$$\hat{\beta}_j = \left\{\frac{-(m-j)}{s}\left[\sum_{i=1}^{s}\left[\log \bar{F}(t_{(ij+1)}) - \log \bar{F}(t_{(ij)})\right] - \left\{(n-s)\left[\log \bar{F}(C,\underline{\theta}) - \log \bar{F}(t_{(sj)},\underline{\theta})\right]\right\}\right]\right\}^{-1}, j = 1,2,\ldots,(m-k). \tag{4}$$

Remark (3): In particular, For 1-out-of-2 subsystem, ($k=1$, $m=2$, $j=(m-k)=(2-1)=1$), we have

$$\hat{\beta}_1 = \left\{\frac{-1}{s}\left[\sum_{i=1}^{s}\left[\log \bar{F}(t_{(i2)}) - \log \bar{F}(t_{(i1)})\right] - \left\{(n-s)\left[\log \bar{F}(C,\underline{\theta}) - \log \bar{F}(t_{(s1)},\underline{\theta})\right]\right\}\right]\right\}^{-1}.$$

For 2-out-of-4 subsystem, ($k=2$, $m=4$, $j=(m-k)=(4-2)=2$, $\Rightarrow j=1,2$), we have

$$\hat{\beta}_1 = \left\{\frac{-3}{s}\left[\sum_{i=1}^{s}\left[\log \bar{F}(t_{(i2)}) - \log \bar{F}(t_{(i1)})\right] - \left\{(n-s)\left[\log \bar{F}(C,\underline{\theta}) - \log \bar{F}(t_{(s1)},\underline{\theta})\right]\right\}\right]\right\}^{-1},$$

$$\hat{\beta}_2 = \left\{\frac{-2}{s}\left[\sum_{i=1}^{s}\left[\log \bar{F}(t_{(i3)}) - \log \bar{F}(t_{(i2)})\right] - \left\{(n-s)\left[\log \bar{F}(C,\underline{\theta}) - \log \bar{F}(t_{(s2)},\underline{\theta})\right]\right\}\right]\right\}^{-1}.$$

In the next subsection, we discuss the parameter estimation procedure with Exponential and Weibull baseline distribution.

3.4.1. Exponential Baseline Distribution:

The pdf and sf of exponential distribution with parameter θ is given by,

$$f(t_{(ij+1)}) = \theta \exp\left(-\theta t_{(ij+1)}\right), \quad t_{(ij+1)} \geq 0, \theta > 0,$$

$$\bar{F}(t_{(ij+1)}) = \exp\left(-\theta t_{(ij+1)}\right), \quad t_{(ij+1)} \geq 0, \theta > 0.$$

The likelihood function corresponding to i^{th} k-out-of-m load sharing subsystem is,

$$L_i = \frac{m!}{(k-1)!}\theta^{m-k+1}\exp(-m\theta t_{(i1)})\prod_{j=1}^{(m-k)}\left\{\beta_j\left[\exp\left(-\theta(t_{(ij+1)} - t_{(ij)})\right)\right]^{(m-j)\beta_j}\right\}, \quad i = 1,2,\ldots,n. \tag{5}$$

Thus likelihood functions under Type-I, Type-II and by combining both the cases are given below.

Likelihood function under Type-I Censoring:

$$L_1 = \left(\frac{m!}{(k-1)!}\right)^p \theta^{p(m-k+1)} \exp\left(-m\theta \sum_{i=1}^{p} t_{(i1)}\right) \prod_{i=1}^{p} \left\{ \prod_{j=1}^{(m-k)} \left\{\beta_j \left[\exp\left(-\theta\left(t_{(ij+1)} - t_{(ij)}\right)\right)\right]^{(m-j)\beta_j}\right\}\right\}$$
$$\left\{\prod_{j=1}^{(m-k)} \left[\exp\left(-\theta(t_{(0j+1)} - t_{(pj)})\right)\right]^{(n-p)(m-j)\beta_j}\right\}.$$

Likelihood function under Type-II Censoring:

$$L_2 = \left(\frac{m!}{(k-1)!}\right)^q \theta^{q(m-k+1)} \exp\left(-m\theta \sum_{i=1}^{q} t_{(i1)}\right) \prod_{i=1}^{q} \left\{ \prod_{j=1}^{(m-k)} \left\{\beta_j \left[\exp\left(-\theta\left(t_{(ij+1)} - t_{(ij)}\right)\right)\right]^{(m-j)\beta_j}\right\}\right\}$$
$$\left\{\prod_{j=1}^{(m-k)} \left[\exp\left(-\theta(t_{(qj+1)} - t_{(qj)})\right)\right]^{(n-q)(m-j)\beta_j}\right\}.$$

Combined likelihood function:

$$L_c = \left(\frac{m!}{(k-1)!}\right)^s \theta^{s(m-k+1)} \exp\left(-m\theta \sum_{i=1}^{s} t_{(i1)}\right) \prod_{i=1}^{s} \left\{ \prod_{j=1}^{(m-k)} \left\{\beta_j \left[\exp\left(-\theta\left(t_{(ij+1)} - t_{(ij)}\right)\right)\right]^{(m-j)\beta_j}\right\}\right\}$$
$$\left\{\prod_{j=1}^{(m-k)} \left[\exp\left(-\theta(C - t_{(sj)})\right)\right]^{(n-s)(m-j)\beta_j}\right\}.$$

The log-likelihood function for combined case is given by.

$$\log L_c = s\log\left(\frac{m!}{(k-1)!}\right) + s(m-k+1)\log\theta - m\theta\sum_{i=1}^{s} t_{(i1)} + \left\{\sum_{i=1}^{s}\sum_{j=1}^{(m-k)} \left\{\log\beta_j - \left[(m-j)\theta\beta_j\left(t_{(ij+1)} - t_{(ij)}\right)\right]\right\}\right\}$$
$$-\left\{\sum_{j=1}^{(m-k)} \left[(n-s)(m-j)\theta\beta_j\left(C - t_{(sj)}\right)\right]\right\}.$$

For obtaining maximum likelihood estimates $(\hat{\theta}, \hat{\beta}_j)$ of (θ, β_j), we have log-likelihood equations as,

$$\frac{\partial \log L_c}{\partial\theta} = \frac{s(m-k+1)}{\theta} - m\sum_{i=1}^{s} t_{(i1)} - \sum_{i=1}^{s}\sum_{j=1}^{(m-k)} \left[(m-j)\beta_j\left(t_{(ij+1)} - t_{(ij)}\right)\right] - \sum_{j=1}^{(m-k)} \left[(n-s)(m-j)\beta_j\left(C - t_{(sj)}\right)\right] = 0, \tag{6}$$

$$\frac{\partial \log L_c}{\partial\beta_j} = \frac{s}{\beta_j} - \theta(m-j)\sum_{i=1}^{s}\left[\left(t_{(ij+1)} - t_{(ij)}\right)\right] - \left[\theta(n-s)(m-j)\left(C - t_{(sj)}\right)\right] = 0, j = 1, 2, \ldots, (m-k). \tag{7}$$

From equations (5) and (6) maximum likelihood estimates $(\hat{\theta}, \hat{\beta}_j)$ of (θ, β_j) are,

$$\hat{\theta} = \left\{\frac{1}{s(m-k+1)}\left[m\sum_{i=1}^{s} t_{(i1)} + \sum_{i=1}^{s}\sum_{j=1}^{(m-k)} \left[(m-j)\beta_j\left(t_{(ij+1)} - t_{(ij)}\right)\right] + \sum_{j=1}^{(m-k)} \left[(m-j)\beta_j(n-s)\left(C - t_{(sj)}\right)\right]\right]\right\}^{-1}, \tag{8}$$

$$\hat{\beta}_j = \left\{ \frac{\theta(m-j)}{s} \left[\sum_{i=1}^{s} \left[\left(t_{(ij+1)} - t_{(ij)} \right) \right] + \left[(n-s) \left(C - t_{(sj)} \right) \right] \right] \right\}^{-1}. \tag{9}$$

Remark (4)

(1) The combined likelihood function in parallel configuration composed by 1-out-of-2 subsystem is

$$\begin{aligned} L_c =& (2!)^s \theta^{2s} \exp\left(-2\theta \sum_{i=1}^{s} t_{(i1)} \right) \prod_{i=1}^{n} \left\{ \beta_1 \left[\exp\left(-\theta\beta_1 \left(t_{(i2)} - t_{(i1)} \right) \right) \right] \right\} \\ & \left[\exp\left(-\theta\beta_1 (n-s)(C - t_{(sj)}) \right) \right]. \end{aligned}$$

(2) The combined likelihood function in parallel configuration composed by 2-out-of-4 subsystem is

$$\begin{aligned} L_c =& (4!)^s \theta^{3s} \exp\left(-4\theta \sum_{i=1}^{s} t_{(i1)} \right) \prod_{i=1}^{s} \left\{ \left[\beta_1 \exp\left(-3\theta\beta_1 \left(t_{(i2)} - t_{(i1)} \right) \right) \right] \right. \\ & \left. \left[\beta_2 \exp\left(-2\theta\beta_2 \left(t_{(i3)} - t_{(i2)} \right) \right) \right] \right\} \quad \exp\left[-3\theta\beta_1 (n-s)(C - t_{(s1)}) \right] \\ & \exp\left[-2\theta\beta_2 (n-s)(C - t_{(s2)}) \right] \end{aligned}$$

3.4.2. Weibull Baseline Distribution:

The pdf and sf of weibull distribution with parameter a (shape parameter), b (scale parameter) are respectively given by,

$$f(t_{(ij+1)}) = \frac{a}{b} \left[\frac{t_{(ij+1)}}{b} \right]^{a-1} \exp\left[-\left(\frac{t_{(ij+1)}}{b} \right)^a \right], \quad t_{(ij+1)} \geq 0, a > 0, b > 0,$$

$$\bar{F}(t_{(ij+1)}) = \exp\left[-\left(\frac{t_{(ij+1)}}{b} \right)^a \right], \quad t_{(ij+1)} \geq 0, a > 0, b > 0.$$

The likelihood based on i^{th} k-out-of-m subsystem is,

$$\begin{aligned} L_i =& \frac{m!}{(k-1)!} \left(\frac{a}{b} \right)^{(m-k+1)} \left(\frac{t_{(i1)}}{b} \right)^{(a-1)} \exp\left[-m \left(\frac{t_{(i1)}}{b} \right)^a \right] \\ & \prod_{j=1}^{(m-k)} \left\{ \beta_j \left(\frac{t_{(ij+1)}}{b} \right)^{(a-1)} \exp\left[\frac{-(m-j)\beta_j}{b^a} \left(t^a_{(ij+1)} - t^a_{(ij)} \right) \right] \right\}. \end{aligned}$$

Thus likelihood functions under Type-I, Type-II and by combining both the cases are given below.

Likelihood function under Type-I Censoring:

$$L_1 = \left(\frac{m!}{(k-1)!}\right)^p \left(\frac{a}{b}\right)^{p(m-k+1)} \exp\left(-m\sum_{i=1}^{p}\left(\frac{t_{(i1)}}{b}\right)^a\right) \prod_{i=1}^{p}\left(\frac{t_{(i1)}}{b}\right)^{a-1} \prod_{i=1}^{p}\prod_{j=1}^{(m-k)}\left\{\beta_j\left(\frac{t_{(ij+1)}}{b}\right)^{(a-1)}\right.$$
$$\left.\exp\left[\frac{-(m-j)\beta_j}{b^a}\left(t^a_{(ij+1)}-t^a_{(ij)}\right)\right]\right\} \prod_{j=1}^{(m-k)}\left\{\exp\left(\frac{-(n-p)(m-j)\beta_j}{b^a}\left(t^a_{(0j+1)}-t^a_{(pj)}\right)\right)\right\}$$

Likelihood function under Type-II Censoring:

$$L_2 = \left(\frac{m!}{(k-1)!}\right)^q \left(\frac{a}{b}\right)^{q(m-k+1)} \exp\left(-m\sum_{i=1}^{q}\left(\frac{t_{(i1)}}{b}\right)^a\right) \prod_{i=1}^{q}\left(\frac{t_{(i1)}}{b}\right)^{a-1} \prod_{i=1}^{q}\prod_{j=1}^{(m-k)}\left\{\beta_j\left(\frac{t_{(ij+1)}}{b}\right)^{(a-1)}\right.$$
$$\left.\exp\left[\frac{-(m-j)\beta_j}{b^a}\left(t^a_{(ij+1)}-t^a_{(ij)}\right)\right]\right\} \prod_{j=1}^{(m-k)}\left\{\exp\left(\frac{-(n-q)(m-j)\beta_j}{b^a}\left(t^a_{(qj+1)}-t^a_{(qj)}\right)\right)\right\}$$

Combined likelihood function:

$$L_c = \left(\frac{m!}{(k-1)!}\right)^s \left(\frac{a}{b}\right)^{s(m-k+1)} \exp\left(-m\sum_{i=1}^{s}\left(\frac{t_{(i1)}}{b}\right)^a\right) \prod_{i=1}^{s}\left(\frac{t_{(i1)}}{b}\right)^{a-1} \prod_{i=1}^{s}\prod_{j=1}^{(m-k)}\left\{\beta_j\left(\frac{t_{(ij+1)}}{b}\right)^{(a-1)}\right.$$
$$\left.\exp\left[\frac{-(m-j)\beta_j}{b^a}\left(t^a_{(ij+1)}-t^a_{(ij)}\right)\right]\right\} \prod_{j=1}^{(m-k)}\left\{\exp\left(\frac{-(n-s)(m-j)\beta_j}{b^a}\left(C^a-t^a_{(sj)}\right)\right)\right\}$$

The log-likelihood function for combined case is given by.

$$\log L_c = s\log\left(\frac{m!}{(k-1)!}\right) + s(m-k+1)\log\left(\frac{a}{b}\right) - m\sum_{i=1}^{n}\left(\frac{t_{(i1)}}{b}\right)^a + (a-1)\sum_{i=1}^{s}\log\left(\frac{t_{(i1)}}{b}\right) + \sum_{i=1}^{s}\sum_{j=1}^{(m-k)}\{\log\beta_j$$
$$+(a-1)\log\left(\frac{t_{(ij+1)}}{b}\right) - \frac{(m-j)\beta_j}{b^a}\left(t^a_{(ij+1)}-t^a_{(ij)}\right)\Big\} - \sum_{j=1}^{(m-k)}\frac{(n-s)(m-j)\beta_j}{b^a}\left(C^a-t^a_{(sj)}\right).$$

For obtaining maximum likelihood estimates $(\hat{a}, \hat{b}, \hat{\beta}_j)$ of (a, b, β_j), we have log-likelihood equations as,

$$\frac{\partial \log L_c}{\partial a} = \frac{s(m-k+1)}{a} + \sum_{i=1}^{s}\log\left(\frac{t_{(i1)}}{b}\right) - m\sum_{i=1}^{s}\left(\frac{t_{(i1)}}{b}\right)^a\log\left(\frac{t_{(i1)}}{b}\right) + \sum_{i=1}^{s}\sum_{j=1}^{(m-k)}\log\left(\frac{t_{(ij+1)}}{b}\right)$$
$$-\sum_{i=1}^{s}\sum_{j=1}^{(m-k)}\left\{(m-j)\beta_j\left[\left(\frac{t_{(ij+1)}}{b}\right)^a\log\left(\frac{t_{(ij+1)}}{b}\right)-\left(\frac{t_{(ij)}}{b}\right)^a\log\left(\frac{t_{(ij)}}{b}\right)\right]\right\}$$
$$-\sum_{j=1}^{(m-k)}\left\{(n-s)(m-j)\beta_j\left[\left(\frac{C}{b}\right)^a\log\left(\frac{C}{b}\right)-\left(\frac{t_{(sj)}}{b}\right)^a\log\left(\frac{t_{(sj)}}{b}\right)\right]\right\} = 0,$$

$$\frac{\partial \log L_c}{\partial b} = \frac{s(m-k+1)}{b} - \frac{s(a-1)}{b} + \frac{am\sum_{i=1}^{s} t^a_{(i1)}}{b^{a+1}} - \frac{s(m-k)(a-1)}{b}$$
$$+\sum_{i=1}^{s}\sum_{j=1}^{(m-k)}\frac{(m-j)\beta_j a}{b^{a+1}}\left(t^a_{(ij+1)}-t^a_{(ij)}\right) + \sum_{j=1}^{(m-k)}\left\{\frac{a(n-s)(m-j)\beta_j}{b^{a+1}}\left(C^a-t^a_{(sj)}\right)\right\} = 0,$$

and,

$$\frac{\partial \log L_c}{\partial \beta_j} = \frac{s}{\beta_j} - \sum_{i=1}^{n} \frac{(m-j)}{b^a}\left(t^a_{(ij+1)} - t^a_{(ij)}\right) - \frac{(n-s)(m-j)}{sb^a}\left(C^a - t^a_{(sj)}\right) = 0. \tag{10}$$

From (9),

$$\hat{\beta}_j = \left\{ \frac{(m-j)}{sb^a}\left[\sum_{i=1}^{s}\left(t^a_{(ij+1)} - t^a_{(ij)}\right) + (n-s)\left(C^a - t^a_{(sj)}\right)\right]\right\}^{-1} \tag{11}$$

Remark (5)

(1) The combined likelihood function in parallel configuration composed by 1-out-of-2 subsystem is

$$L_c = (2!)^s \left(\frac{a}{b}\right)^{2s} \prod_{i=1}^{s}\left(\frac{t_{(i1)}}{b}\right)^{(a-1)} \exp\left[-2\sum_{i=1}^{s}\left(\frac{t_{(i1)}}{b}\right)^a\right] \prod_{i=1}^{s}\left\{\beta_1\left(\frac{t_{(i2)}}{b}\right)^{(a-1)} \exp\left[\frac{-\beta_1}{b^a}\left(t^a_{(i2)} - t^a_{(i1)}\right)\right]\right\} \exp\left[\frac{-(n-s)\beta_1}{b^a}\left(C^a - t^a_{(s1)}\right)\right].$$

(2) The combined likelihood function in parallel configuration composed by 2-out-of-4 subsystem is

$$L_c = (4!)^n \left(\frac{a}{b}\right)^{3n} \prod_{i=1}^{n}\left(\frac{t_{(i1)}}{b}\right)^{(a-1)} \exp\left[-4\sum_{i=1}^{n}\left(\frac{t_{(i1)}}{b}\right)^a\right] \prod_{i=1}^{n}\left\{\beta_1\left(\frac{t_{(i2)}}{b}\right)^{(a-1)} \exp\left[\frac{-3\beta_1}{b^a}\left(t^a_{(i2)} - t^a_{(i1)}\right)\right] \beta_2\left(\frac{t_{(i3)}}{b}\right)^{(a-1)} \exp\left[\frac{-2\beta_2}{b^a}\left(t^a_{(i3)} - t^a_{(i2)}\right)\right]\right\} \exp\left[\frac{-3\beta_1(n-s)}{b^a}\left(C^a - t^a_{(s1)}\right)\right] \exp\left[\frac{-2\beta_2(n-s)}{b^a}\left(C^a - t^a_{(s2)}\right)\right].$$

4. Simulation

In this section, we assess model parameters, accompanied by their standard errors, through the maximum likelihood estimation procedure under Type-I and Type-II censoring. This evaluation is conducted in the context of a load-sharing parallel configuration with "1-out-of-2" and "2-out-of-4" subsystems with the baseline distribution of each component follow either exponential or Weibull distribution.

4.1. Type-I Censoring

We conducted simulations for Type-I censoring, employing three termination times. To determine these termination times, we utilized the 95^{th} quantile and employed a trial-and-error approach to identify suitable termination points.

4.1.1. Exponential distribution

We conducted a simulation study using a baseline exponential distribution with a rate parameter ($\theta = 2$) with ($n = 30, 100, 500$). This study was repeated for two subsystem configurations: "1-out-of-2" and "2-out-of-4". For the "1-out-of-2" subsystem, we considered ($\beta = 0.2, 0.5, 1, 1.5, 2$). For the "2-out-of-4" subsystem, we used ($\beta_1, \beta_2 = 0.5, 1, 2$). We calculated the average parameter estimates for (θ) and ($\beta_j, j = 1, 2$), along with their standard errors (SE) based on 1000 repetitions, for three different termination/censoring times: ($t_0 = 2.9, 1.5, 0.7$) for the "1-out-of-2" case and ($t_0 = 1.9, 0.8, 0.4$) for the "2-out-of-4" case, using the Type-I censoring approach. The results related to "1-out-of-2" case and "2-out-of-4" case are presented in Tables 1 and 2 respectively.

From Table 1, we observe that as the censoring time (t_0) decreases (resulting in more censored observations), both the bias and the variability of the estimates for (θ) increase. The estimates for the load sharing parameter (β) exhibit similar behavior. Additionally, as (β) increases, its bias as well as variability also increases. However, as the number of parallel subsystems (n) increases, the bias and variability in the estimates decrease.

Table 2 shows that a decrease in (t_0) leads to increased bias and variability in the estimates for both θ and β. While the bias and variability of (θ) decrease as (n) increases, the estimates for (β) are not significantly affected by different values of ($n = 30, 100, 500$).

Table 1 Parameter estimates for different termination times $t = 0.7, 1.5, 2.9$,baseline parameter $\theta = 2$ and different values of n for 1-out-of-2 subsystem with Exponential baseline distribution.

		t=2.9				t=1.5				t=0.7			
n	β	$\hat{\theta}$	$\hat{\beta}$	SE of θ	SE of β	$\hat{\theta}$	$\hat{\beta}$	SE of θ	SE of β	$\hat{\theta}$	$\hat{\beta}$	SE of θ	SE of β
30	0.2	2.2707	0.1931	0.5687	0.0699	2.5975	0.1796	0.8930	0.0944	3.5246	0.1575	1.9554	0.1607
30	0.5	2.1614	0.4972	0.4372	0.1442	2.4059	0.4522	0.5254	0.1740	3.2470	0.6586	0.7475	0.3499
30	1	2.0880	1.0252	0.3773	0.2673	2.2513	0.9647	0.4294	0.2998	3.2470	0.6586	0.7475	0.3499
30	1.5	2.0603	1.5582	0.4024	0.4225	2.2059	1.4508	0.4209	0.4085	3.0986	1.0064	0.5976	0.4742
30	2	2.0842	2.0749	0.4008	0.5610	2.1623	1.9567	0.3971	0.5530	2.9926	1.3280	0.5465	0.5156
100	0.2	2.1515	0.1888	0.2690	0.0360	2.4605	0.1776	0.3940	0.0731	3.4955	0.1392	0.8040	0.1252
100	0.5	2.0808	0.4895	0.2144	0.0753	2.3254	0.4482	0.2669	0.1129	3.3107	0.3333	0.5184	0.2050
100	1	2.0204	1.0100	0.2053	0.1465	2.2231	0.9218	0.2240	0.1667	3.1239	0.6585	0.3547	0.2760
100	1.5	2.0140	1.5207	0.2073	0.2274	2.1474	1.4103	0.2091	0.2261	3.0288	0.9812	0.3101	0.3307
100	2	2.0150	2.0170	0.1940	0.2885	2.1161	1.9244	0.2096	0.2998	2.9178	1.3191	0.2932	0.3899
500	0.2	2.1355	0.1892	0.1161	0.0201	2.4197	0.1737	0.1676	0.0513	3.3718	0.1375	0.3301	0.1088
500	0.5	2.0613	0.4880	0.0959	0.0353	2.3084	0.4456	0.1177	0.0826	3.2543	0.3306	0.2180	0.1796
500	1	2.0180	0.9978	0.0882	0.0659	2.2005	0.9085	0.0988	0.0982	3.1025	0.6508	0.1551	0.2455
500	1.5	2.0083	1.5029	0.0869	0.0948	2.1383	1.3953	0.0948	0.1188	2.9967	0.9720	0.1331	0.2846
500	2	2.0057	1.9955	0.0867	0.1267	2.1001	1.8955	0.0908	0.1363	2.9007	1.2994	0.1274	0.3158

Table 2 Parameter estimates for different termination times $t = 0.4, 0.8, 1.9$,baseline parameter $\theta = 2$ and different values of n for 2-out-of-4 subsystem with Exponential baseline distribution.

			t=1.9						t=0.8						t=0.4					
n	β_1	β_2	$\hat{\theta}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of θ	SE of β_1	SE of β_2	$\hat{\theta}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of θ	SE of β_1	SE of β_2	$\hat{\theta}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of θ	SE of β_1	SE of β_2
30	0.5	0.5	2.1268	0.3726	0.4980	0.4347	0.1396	0.1410	2.4733	0.2400	0.4339	0.6465	0.1349	0.1910	3.4651	0.1712	0.3326	1.3527	0.1678	0.2550
30	0.5	1	2.1142	0.4551	0.9728	0.4251	0.1441	0.2780	2.3932	0.3410	0.8733	0.5272	0.1603	0.3538	3.2634	0.2525	0.6725	0.9941	0.1967	0.4466
30	0.5	2	2.0853	0.4893	1.9304	0.3927	0.1344	0.5348	2.3382	0.4113	1.7226	0.5053	0.1609	0.6425	3.1878	0.2914	1.2140	0.9355	0.2103	0.7845
100	0.5	0.5	2.0740	0.3491	0.4896	0.2147	0.0717	0.0774	2.3597	0.2308	0.4375	0.3025	0.0851	0.1188	3.1743	0.1828	0.3650	0.5478	0.1651	0.2396
100	0.5	1	2.0564	0.4450	0.9751	0.2112	0.0765	0.1473	2.3096	0.3308	0.8879	0.2729	0.1060	0.2270	3.0923	0.2350	0.6768	0.4533	0.1580	0.3591
100	0.5	2	2.0423	0.4811	1.9427	0.2064	0.0769	0.3010	2.2710	0.3922	1.7539	0.2502	0.0989	0.3933	3.0294	0.2947	1.3471	0.4077	0.1773	0.6521
500	0.5	0.5	2.0478	0.3456	0.4914	0.1000	0.0362	0.0377	2.3341	0.2312	0.4409	0.1329	0.0782	0.0965	3.0781	0.1687	0.3514	0.2198	0.1345	0.1856
500	0.5	1	2.0341	0.4443	0.9860	0.0945	0.0342	0.0652	2.2937	0.3234	0.8856	0.1219	0.0777	0.1570	3.0267	0.2379	0.7029	0.1915	0.1638	0.3542
500	0.5	2	2.0182	0.4823	1.9792	0.0909	0.0336	0.1309	2.2580	0.3905	1.7914	0.1108	0.0697	0.2666	2.9882	0.2920	1.4089	0.1735	0.1590	0.6321
30	1	0.5	2.1146	0.6910	0.5028	0.4201	0.3235	0.1455	2.4200	0.3603	0.4467	0.5503	0.2115	0.1845	3.3044	0.2479	0.3618	1.1299	0.2615	0.2729
30	1	1	2.0764	0.9607	1.0176	0.4011	0.2848	0.2625	2.3130	0.6161	0.9182	0.4622	0.2753	0.3191	3.1482	0.3895	0.6921	0.7836	0.3073	0.4066
30	1	2	2.0529	1.0407	2.0852	0.3919	0.2921	0.5810	2.2650	0.8187	1.8308	0.4200	0.2970	0.5780	3.0394	0.5425	1.3758	0.6827	0.3471	0.7353
100	1	0.5	2.0461	0.6422	0.4977	0.2131	0.1487	0.0734	2.3313	0.3463	0.4476	0.2926	0.1368	0.1159	3.1033	0.2415	0.3537	0.4843	0.2502	0.2058
100	1	1	2.0311	0.9282	1.0011	0.2017	0.1573	0.1414	2.2683	0.5907	0.9023	0.2456	0.1938	0.1964	3.0236	0.3945	0.7132	0.3944	0.2795	0.3392
100	1	2	2.0351	0.9924	1.9965	0.2107	0.1462	0.2865	2.2072	0.7894	1.8305	0.2249	0.1803	0.3382	2.9363	0.5268	1.3959	0.3494	0.2620	0.5278
500	1	0.5	2.0404	0.6186	0.4920	0.0959	0.0702	0.0338	2.3077	0.3366	0.4412	0.1271	0.1150	0.0795	3.0513	0.2422	0.3586	0.2131	0.2254	0.1879
500	1	1	2.0139	0.9164	0.9981	0.0911	0.0729	0.0645	2.2412	0.5825	0.9048	0.1091	0.1440	0.1318	2.9705	0.3758	0.6970	0.1694	0.2522	0.2902
500	1	2	2.0073	0.9890	1.9968	0.0884	0.0654	0.1275	2.1874	0.7773	1.8343	0.0985	0.1181	0.2003	2.9083	0.5151	1.4009	0.1460	0.2347	0.4648
30	2	0.5	2.1006	1.2940	0.5091	0.4121	0.7530	0.1480	2.3732	0.4951	0.4618	0.5380	0.2991	0.1781	3.3059	0.3071	0.3621	1.0324	0.3515	0.2604
30	2	1	2.0846	1.9827	1.0287	0.4128	0.6392	0.2913	2.3061	1.0405	0.9078	0.4744	0.5411	0.3016	3.0452	0.5738	0.7089	0.7182	0.4626	0.3671
30	2	2	2.0741	2.0542	2.0569	0.3890	0.5515	0.5488	2.2001	1.6451	1.8943	0.4060	0.6322	0.5808	2.9521	0.9059	1.3867	0.5736	0.5245	0.6032
100	2	0.5	2.0409	1.1173	0.5009	0.2140	0.3328	0.0735	2.3161	0.4651	0.4487	0.2608	0.2323	0.1084	3.0633	0.3183	0.3677	0.4559	0.3614	0.2109
100	2	1	2.0286	1.8829	1.0037	0.2006	0.3392	0.1372	2.2317	1.0100	0.9180	0.2435	0.3538	0.1826	2.9782	0.5673	0.7022	0.3597	0.4371	0.2953
100	2	2	2.0168	2.0290	2.0338	0.2084	0.3024	0.2992	2.1503	1.5941	1.8860	0.2143	0.3695	0.3257	2.8532	0.8909	1.3933	0.2992	0.4502	0.4548
500	2	0.5	2.0316	1.0523	0.4955	0.0938	0.1502	0.0337	2.2955	0.4588	0.4459	0.1220	0.2230	0.0821	3.0260	0.3134	0.3579	0.1971	0.4015	0.1840
500	2	1	2.0097	1.8569	0.9977	0.0883	0.1591	0.0629	2.2104	0.9874	0.9129	0.0997	0.2840	0.1125	2.9342	0.5746	0.7057	0.1529	0.4438	0.2620
500	2	2	2.0060	1.9970	2.0026	0.0883	0.1258	0.1242	2.1345	1.5735	1.8816	0.0939	0.2305	0.1678	2.8218	0.8900	1.3993	0.1329	0.4210	0.3896

4.1.2. Weibull distribution

We have taken baseline distribution as Weibull (a, b). To asses the performances of estimates for two different configurations composed by "1-out-of-2" and "2-out-of-4" subsystems, we have conducted a simulation study.

For the "1-out-of-2" subsystem under a Type-I censoring approach, we considered $n = 30, 100, 500$ and various parameter combinations: $a = 0.5, b = 2$ with $\beta = 0.5, 1, 2$ at different censoring times $t_0 = 7, 20, 40$; and $a = 2, b = 0.5$ with $\beta = 0.5, 1, 2$ at censoring times $t_0 = 0.5, 0.9, 1.5$. The average parameter estimates for a, b, and β, along with their standard errors (SE) based on 1000 repetitions, are presented in Table 3.

For the "2-out-of-4" subsystem under the Type-I censoring approach, we used $n = 30, 100, 500$ and parameter combinations: $a = 0.5, b = 2$ with $\beta_1, \beta_2 = 0.5, 1, 2$ at censoring times $t_0 = 7, 14, 20$; and $a = 2, b = 0.5$ with $\beta_1, \beta_2 = 0.5, 1, 2$ at censoring times $t_0 = 0.5, 0.7, 1.5$. The average parameter estimates for a, b, and β_j (for $j = 1, 2$), along with their standard errors (SE) based on 1000 repetitions, are shown in Table 4.

From Table 3, we observe that as the censoring time increases, the bias in the estimates of the shape parameter a decreases. Similarly, as the sample size n increases, the bias in the estimates of a decreases. However, the bias in the estimates of a increases with higher values of a. Additionally, the termination/censoring time t_0 and the shape parameter a affect the scale parameter b; the bias in b decreases with increasing t_0 and a. However, the bias in b increases as b increases. For the load-sharing parameter β, its bias decreases with increasing t_0 and b, but increases with higher values of a and β.

Similar conclusions can be drawn from Table 4 for the "2-out-of-4" subsystem under the Type-I censoring approach.

Table 3 Parameter estimates for different termination times with different values of baseline parameters and n for 1-out-of-2 subsystem with Weibull baseline distribution.

				$t_0=7$						$t_0=20$						$t_0=40$					
n	a	b	β	$\hat{a}$	$\hat{b}$	$\hat{\beta}$	SE of a	SE of b	SE of β	$\hat{a}$	$\hat{b}$	$\hat{\beta}$	SE of a	SE of b	SE of β	$\hat{a}$	$\hat{b}$	$\hat{\beta}$	SE of a	SE of b	SE of β
30	0.5	2	0.5	0.5563	1.0543	0.3694	0.1189	0.4996	0.2590	0.5289	1.5582	0.4449	0.0790	0.6618	0.2119	0.5256	1.7935	0.4603	0.0694	0.6970	0.1825
30	0.5	2	1	0.5503	1.1501	0.7258	0.0902	0.4236	0.3383	0.5271	1.7256	0.9258	0.0706	0.6377	0.3326	0.5200	1.9049	0.9824	0.0682	0.6939	0.3307
30	0.5	2	2	0.5444	1.2735	1.4570	0.0705	0.3910	0.4869	0.5259	1.8914	1.9112	0.0668	0.6551	0.5648	0.5236	1.9798	1.9451	0.0638	0.7654	0.5878
100	0.5	2	0.5	0.5324	1.0345	0.3683	0.0656	0.2626	0.1810	0.5142	1.5328	0.4390	0.0451	0.3616	0.1258	0.5100	1.8060	0.4722	0.0379	0.3837	0.1047
100	0.5	2	1	0.5316	1.1255	0.7181	0.0473	0.2239	0.2233	0.5163	1.6844	0.8946	0.0365	0.3361	0.1777	0.5076	1.9132	0.9650	0.0333	0.3955	0.1690
100	0.5	2	2	0.5312	1.2813	1.4912	0.0400	0.2262	0.3143	0.5110	1.8311	1.8866	0.0341	0.3506	0.2992	0.5067	1.9989	2.0026	0.0330	0.3963	0.3093
500	0.5	2	0.5	0.5242	1.0249	0.3587	0.0384	0.1223	0.1388	0.5108	1.5431	0.4357	0.0242	0.1570	0.0843	0.5057	1.7835	0.4676	0.0181	0.1690	0.0565
500	0.5	2	1	0.5292	1.1235	0.7157	0.0285	0.1045	0.1765	0.5117	1.6892	0.8976	0.0174	0.1482	0.0945	0.5046	1.9119	0.9706	0.0147	0.1655	0.0771
500	0.5	2	2	0.5272	1.2785	1.4711	0.0214	0.0969	0.2021	0.5077	1.8420	1.8876	0.0147	0.1600	0.1399	0.5021	1.9783	1.9832	0.0144	0.1789	0.1391
				$t_0=0.5$						$t_0=0.9$						$t_0=1.5$					
30	2	0.5	0.5	2.6498	0.3362	0.2294	1.1901	0.0566	0.2802	2.2503	0.4193	0.3628	0.4589	0.0517	0.2552	2.0796	0.4932	0.4910	0.2686	0.0456	0.1704
30	2	0.5	1	2.3848	0.3484	0.4584	0.6387	0.0408	0.3711	2.1978	0.4339	0.7309	0.3515	0.0401	0.3584	2.0970	0.4955	0.9907	0.2699	0.0470	0.3197
30	2	0.5	2	2.3138	0.3587	0.8707	0.4354	0.0322	0.4868	2.1493	0.4487	1.5599	0.2873	0.0354	0.5321	2.0815	0.4935	1.9825	0.2611	0.0463	0.6060
100	2	0.5	0.5	2.2991	0.3451	0.2407	0.4593	0.0298	0.2106	2.1256	0.4249	0.3709	0.2429	0.0270	0.1776	2.0291	0.4973	0.4925	0.1384	0.0248	0.0929
100	2	0.5	1	2.2474	0.3509	0.4578	0.3181	0.0218	0.2885	2.1189	0.4351	0.7363	0.1777	0.0221	0.2235	2.0344	0.4969	0.9848	0.1362	0.0253	0.1678
100	2	0.5	2	2.1952	0.3620	0.8982	0.2451	0.0168	0.3684	2.1061	0.4513	1.5392	0.1595	0.0199	0.3194	2.0315	0.4991	1.9966	0.1409	0.0256	0.3048
500	2	0.5	0.5	2.1787	0.3477	0.2554	0.2544	0.0135	0.1947	2.0913	0.4275	0.3661	0.1474	0.0120	0.1341	2.0061	0.4981	0.4947	0.0592	0.0113	0.0413
500	2	0.5	1	2.1916	0.3537	0.4790	0.1936	0.0098	0.2519	2.1091	0.4371	0.7334	0.1026	0.0103	0.1719	2.0061	0.4995	0.9986	0.0586	0.0117	0.0747
500	2	0.5	2	2.2013	0.3627	0.8915	0.1540	0.0077	0.3185	2.0983	0.4520	1.5147	0.0821	0.0088	0.1934	2.0036	0.4992	2.0006	0.0591	0.0113	0.1351

Table 4 Parameter estimates for different termination times with different values of baseline parameters and n for 2-out-of-4 subsystem with Weibull baseline distribution.

					$t_0=7$								$t_0=14$								$t_0=20$							
n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of a	SE of b	SE of β_1	SE of β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of a	SE of b	SE of β_1	SE of β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of a	SE of b	SE of β_1	SE of β_2
30	0.5	2	0.5	0.5	0.4423	2.2665	0.2497	0.5625	0.0741	1.5309	0.1221	0.3140	0.4433	2.5357	0.3527	0.6316	0.0595	1.3469	0.1403	0.2693	0.4567	2.4484	0.3897	0.6047	0.0596	1.2135	0.1300	0.2251
30	0.5	2	1	0.5	0.4150	2.7063	0.4620	0.6849	0.0585	1.5375	0.2021	0.3020	0.4216	2.9511	0.6211	0.7020	0.0560	1.5541	0.2309	0.2462	0.4413	2.8030	0.7543	0.6737	0.0643	1.4002	0.2305	0.2161
30	0.5	2	2	0.5	0.3770	3.7912	0.8111	0.8085	0.0504	2.0747	0.3329	0.2721	0.3855	3.8148	1.0552	0.7977	0.0524	2.0620	0.3659	0.2362	0.4116	3.5599	1.3833	0.7534	0.0717	1.9375	0.4719	0.2267
30	0.5	2	0.5	1	0.4700	1.9763	0.3706	0.9121	0.0681	1.0346	0.1470	0.3929	0.4807	2.1073	0.4376	0.9874	0.0634	1.0089	0.1494	0.3594	0.4993	2.0771	0.4758	1.0096	0.0659	0.8990	0.1507	0.3672
30	0.5	2	1	1	0.4698	2.0991	0.7398	1.0671	0.0620	0.9714	0.2499	0.3555	0.4898	2.1325	0.8821	1.0600	0.0658	0.9700	0.2776	0.3427	0.5119	2.0557	0.9723	1.0044	0.0635	0.8809	0.2838	0.3339
30	0.5	2	2	1	0.4498	2.4376	1.4444	1.1612	0.0629	1.1087	0.4641	0.3455	0.4948	2.2633	1.8367	1.0673	0.0784	1.1151	0.5406	0.3401	0.5129	2.0716	1.9397	1.0129	0.0684	1.0140	0.5567	0.3342
30	0.5	2	0.5	2	0.4847	1.9450	0.4610	1.6929	0.0649	0.9033	0.1664	0.6374	0.4982	2.0499	0.4888	1.8513	0.0643	0.9253	0.1579	0.5920	0.5065	2.0642	0.5043	1.9224	0.0644	0.9374	0.1595	0.6206
30	0.5	2	1	2	0.5061	1.8770	0.9151	1.8449	0.0668	0.7890	0.2806	0.5982	0.5145	1.9901	0.9807	1.9678	0.0632	0.8528	0.2906	0.6353	0.5177	1.9733	0.9964	1.9757	0.0665	0.8654	0.3140	0.6480
30	0.5	2	2	2	0.5061	1.8770	0.9151	1.8449	0.0668	0.7890	0.2806	0.5982	0.5182	1.9898	1.9788	1.9807	0.0641	0.8454	0.5452	0.6148	0.5170	2.0072	2.0195	1.9831	0.0600	0.8689	0.5898	0.6021
100	0.5	2	0.5	0.5	0.4277	2.1773	0.2522	0.5799	0.0448	0.8069	0.0899	0.2500	0.4340	2.3906	0.3230	0.6275	0.0369	0.7146	0.0824	0.1936	0.4442	2.4389	0.3923	0.6289	0.0335	0.6815	0.0818	0.1504
100	0.5	2	1	0.5	0.4039	2.6691	0.4676	0.6931	0.0370	0.8489	0.1429	0.2201	0.4128	2.8429	0.5970	0.7132	0.0319	0.8025	0.1350	0.1632	0.4242	2.7891	0.7466	0.6910	0.0305	0.7594	0.1340	0.1221
100	0.5	2	2	0.5	0.3717	3.6065	0.7810	0.7958	0.0333	1.2205	0.2473	0.2051	0.3747	3.8216	1.0364	0.8171	0.0279	1.0668	0.2450	0.1439	0.3929	3.5707	1.3557	0.7942	0.0322	1.0479	0.2504	0.1267
100	0.5	2	0.5	1	0.4584	1.9408	0.3742	0.9571	0.0391	0.5666	0.0916	0.2822	0.4725	2.0527	0.4372	1.0101	0.0350	0.5120	0.0847	0.2245	0.4853	2.0567	0.4790	1.0113	0.0327	0.4973	0.0857	0.1922
100	0.5	2	1	1	0.4549	2.0935	0.7366	1.0757	0.0330	0.5113	0.1519	0.2269	0.4763	2.1330	0.8847	1.0739	0.0332	0.5111	0.1482	0.1920	0.4927	2.0609	0.9604	1.0288	0.0342	0.4915	0.1479	0.1735
100	0.5	2	2	1	0.4389	2.4426	1.4293	1.1842	0.0334	0.6117	0.2761	0.2016	0.4631	2.3184	1.7195	1.1287	0.0392	0.5764	0.2771	0.1787	0.4878	2.1520	1.9343	1.0590	0.0397	0.5486	0.2926	0.1871
100	0.5	2	0.5	2	0.4791	1.8603	0.4534	1.7462	0.0365	0.4516	0.0920	0.3905	0.4878	1.9961	0.4881	1.8983	0.0347	0.5013	0.0862	0.3506	0.4957	2.0302	0.5003	1.9733	0.0326	0.4977	0.0873	0.3709
100	0.5	2	1	2	0.4932	1.8855	0.9166	1.9047	0.0337	0.4251	0.1547	0.3241	0.5030	1.9867	0.9832	1.9745	0.0329	0.4745	0.1589	0.3466	0.5055	1.9679	0.9871	1.9809	0.0310	0.4478	0.1541	0.3365
100	0.5	2	2	2	0.4951	1.9748	1.8694	1.9789	0.0349	0.4445	0.2760	0.3143	0.5057	2.0091	1.9957	2.0115	0.0308	0.4533	0.2879	0.3168	0.5068	1.9848	1.9915	1.9896	0.0334	0.4772	0.3042	0.3245
500	0.5	2	0.5	0.5	0.4228	2.1756	0.2560	0.5955	0.0321	0.5095	0.0723	0.2257	0.4308	2.3311	0.3216	0.6259	0.0225	0.3684	0.0586	0.1603	0.4417	2.3951	0.3897	0.6247	0.0174	0.3148	0.0469	0.1003
500	0.5	2	1	0.5	0.4024	2.6486	0.4623	0.6943	0.0268	0.5433	0.1157	0.1931	0.4069	2.8523	0.6056	0.7176	0.0184	0.4189	0.1000	0.1167	0.4192	2.8128	0.7495	0.7012	0.0149	0.3498	0.0794	0.0706
500	0.5	2	2	0.5	0.3665	3.6326	0.7897	0.8112	0.0230	0.7910	0.2084	0.1684	0.3728	3.7902	1.0447	0.8277	0.0172	0.5986	0.1865	0.1066	0.4192	2.8128	0.7495	0.7012	0.0149	0.3498	0.0794	0.0706
500	0.5	2	0.5	1	0.4562	1.8900	0.3720	0.9532	0.0247	0.2863	0.0677	0.2251	0.4690	2.0342	0.4351	1.0145	0.0181	0.2545	0.0487	0.1455	0.4832	2.0415	0.4742	1.0240	0.0156	0.2250	0.0396	0.0989
500	0.5	2	1	1	0.4533	2.0812	0.7364	1.0933	0.0177	0.2452	0.0945	0.1474	0.4703	2.1382	0.8789	1.0776	0.0156	0.2237	0.0720	0.0936	0.4867	2.0812	0.9623	1.0403	0.0153	0.2186	0.0673	0.0828
500	0.5	2	2	1	0.4342	2.4246	1.4203	1.1889	0.0178	0.2953	0.1753	0.1066	0.4556	2.3379	1.7273	1.1423	0.0164	0.2578	0.1388	0.0842	0.4817	2.1438	1.9063	1.0623	0.0181	0.2520	0.1348	0.0867
500	0.5	2	0.5	2	0.4751	1.8480	0.4513	1.7629	0.0222	0.2305	0.0612	0.2673	0.4869	1.9599	0.4830	1.9088	0.0180	0.2245	0.0469	0.1719	0.4983	2.0030	0.4967	1.9733	0.0157	0.2122	0.0394	0.1561
500	0.5	2	1	2	0.4894	1.8846	0.9088	1.9191	0.0163	0.1914	0.0716	0.1637	0.5000	2.0001	0.9839	1.9904	0.0226	0.3354	0.1106	0.2479	0.5004	1.9929	0.9965	1.9919	0.0130	0.2024	0.0709	0.1481
500	0.5	2	2	2	0.4916	1.9637	1.8635	1.9908	0.0162	0.2039	0.1300	0.1431	0.5001	1.9887	1.9843	1.9946	0.0142	0.2075	0.1358	0.1509	0.5007	2.0109	2.0009	2.0084	0.0141	0.2096	0.1370	0.1522
					$t_0=0.5$								$t_0=0.7$								$t_0=1.5$							
30	2	0.5	0.5	0.5	1.7768	0.4753	0.1875	0.5006	0.3709	0.0836	0.1255	0.3705	1.7575	0.5022	0.2642	0.5824	0.2816	0.0732	0.1363	0.3121	2.0965	0.4911	0.4916	0.4854	0.2453	0.0520	0.1569	0.1873
30	2	0.5	1	0.5	1.6858	0.5020	0.3476	0.6290	0.2880	0.0782	0.2036	0.3795	1.6654	0.5285	0.4836	0.6878	0.2387	0.0701	0.1992	0.2853	2.0800	0.4924	1.0106	0.4948	0.2386	0.0530	0.3185	0.1797
30	2	0.5	2	0.5	1.5470	0.5380	0.5721	0.7194	0.2520	0.0770	0.3115	0.3473	1.5344	0.5699	0.8445	0.7973	0.2138	0.0730	0.3628	0.2775	2.0733	0.4934	2.0270	0.4992	0.2371	0.0520	0.5886	0.1787
30	2	0.5	0.5	1	1.8420	0.4716	0.3032	0.8480	0.3046	0.0666	0.1523	0.4364	1.9042	0.4864	0.3801	0.9466	0.2671	0.0583	0.1436	0.4061	2.0767	0.4914	0.4986	0.9807	0.2383	0.0514	0.1647	0.3457
30	2	0.5	1	1	1.8266	0.4822	0.5706	0.9965	0.2528	0.0567	0.2336	0.4034	1.8968	0.4946	0.7549	1.0477	0.2494	0.0546	0.2489	0.3670	2.0853	0.4900	0.9897	0.9805	0.2402	0.0515	0.2968	0.3417
30	2	0.5	2	1	1.7401	0.5080	1.1002	1.1441	0.2374	0.0571	0.4241	0.3894	1.8145	0.5193	1.5199	1.1844	0.2709	0.0593	0.4877	0.3591	2.0862	0.4913	2.0132	0.9766	0.2372	0.0501	0.5494	0.3098
30	2	0.5	0.5	2	1.9285	0.4651	0.3878	1.4327	0.2973	0.0553	0.1691	0.6078	1.9442	0.4886	0.4732	1.7446	0.2545	0.0550	0.1714	0.6220	2.0760	0.4901	0.4934	1.9750	0.2340	0.0529	0.1580	0.7068
30	2	0.5	1	2	1.9783	0.4730	0.7727	1.7080	0.2576	0.0459	0.2469	0.5813	2.0218	0.4869	0.9328	1.9248	0.2555	0.0510	0.2879	0.6107	2.0762	0.4931	1.0012	1.9912	0.2433	0.0516	0.2786	0.6395
30	2	0.5	2	2	1.9700	0.4857	1.6495	1.9133	0.2687	0.0478	0.5135	0.5754	2.0446	0.4890	1.8941	1.9570	0.2697	0.0507	0.5147	0.5959	2.0873	0.4919	2.0084	1.9725	0.2459	0.0533	0.5637	0.6067
100	2	0.5	0.5	0.5	1.6937	0.4849	0.1970	0.5325	0.2137	0.0504	0.0977	0.2955	1.7121	0.5070	0.2639	0.6040	0.1669	0.0401	0.0885	0.2482	2.0173	0.4985	0.4998	0.5004	0.1270	0.0291	0.0869	0.1047
100	2	0.5	1	0.5	1.6308	0.5053	0.3436	0.6333	0.1908	0.0478	0.1490	0.2871	1.6302	0.5326	0.4824	0.6862	0.1422	0.0402	0.1469	0.2155	2.0209	0.4985	0.9994	0.4966	0.1338	0.0293	0.1574	0.0996
100	2	0.5	2	0.5	1.4961	0.5500	0.5973	0.7504	0.1583	0.0496	0.2514	0.2644	1.4822	0.5777	0.8300	0.8027	0.1245	0.0448	0.2560	0.1902	2.0181	0.4994	2.0038	0.5037	0.1364	0.0309	0.2977	0.1020
100	2	0.5	0.5	1	1.7897	0.4753	0.3015	0.8672	0.1857	0.0389	0.1089	0.3441	1.8449	0.4938	0.3841	0.9743	0.1516	0.0327	0.0917	0.2707	2.0229	0.4970	0.4979	0.9940	0.1201	0.0281	0.0853	0.1899
100	2	0.5	1	1	1.7702	0.4883	0.5679	1.0097	0.1512	0.0325	0.1586	0.2946	1.8437	0.5030	0.7639	1.0781	0.1383	0.0297	0.1413	0.2144	2.0290	0.4970	0.9982	0.9872	0.1201	0.0284	0.1545	0.1772
100	2	0.5	2	1	1.6981	0.5128	1.0831	1.1545	0.1366	0.0321	0.2889	0.2598	1.7657	0.5216	1.4824	1.1786	0.1337	0.0324	0.2735	0.1941	2.0291	0.4974	1.9990	0.9954	0.1264	0.0286	0.2960	0.1767
100	2	0.5	0.5	2	1.8768	0.4723	0.3944	1.5581	0.1680	0.0314	0.1082	0.4699	1.9156	0.4910	0.4632	1.7714	0.1497	0.0294	0.0970	0.3729	2.0265	0.4965	0.4961	1.9783	0.1259	0.0285	0.0883	0.3688
100	2	0.5	1	2	1.9326	0.4759	0.7878	1.7654	0.1393	0.0263	0.1586	0.3519	1.9794	0.4917	0.9277	1.9186	0.1370	0.0277	0.1575	0.3233	2.0216	0.4979	1.0022	2.0033	0.1272	0.0283	0.1595	0.3566
100	2	0.5	2	2	1.9254	0.4873	1.6070	1.9131	0.1389	0.0261	0.2772	0.2977	1.9957	0.4957	1.8997	1.9849	0.1363	0.0282	0.2928	0.3184	2.0313	0.4975	1.9985	2.0064	0.1269	0.0283	0.3098	0.3433
500	2	0.5	0.5	0.5	1.6864	0.4844	0.1913	0.5303	0.1608	0.0324	0.0772	0.2806	1.6974	0.5093	0.2668	0.6088	0.1205	0.0254	0.0697	0.2155	1.9977	0.5007	0.5003	0.5035	0.0549	0.0129	0.0377	0.0446
500	2	0.5	1	0.5	1.6130	0.5096	0.3486	0.6543	0.1388	0.0312	0.1260	0.2621	1.6093	0.5367	0.4889	0.7066	0.0983	0.0251	0.1165	0.1770	1.9953	0.5010	1.0025	0.5052	0.0598	0.0131	0.0684	0.0451
500	2	0.5	2	0.5	1.4829	0.5520	0.5949	0.7530	0.1308	0.0369	0.2280	0.2408	1.4773	0.5770	0.8258	0.8056	0.0921	0.0294	0.2169	0.1578	1.9932	0.5014	1.9971	0.5061	0.0697	0.0144	0.1311	0.0480
500	2	0.5	0.5	1	1.7905	0.4748	0.2923	0.8467	0.1290	0.0225	0.0835	0.3115	1.8320	0.4948	0.3862	0.9704	0.0889	0.0173	0.0673	0.2172	2.0037	0.4999	0.4998	0.9989	0.0530	0.0128	0.0379	0.0836
500	2	0.5	1	1	1.7658	0.4885	0.5737	1.0312	0.0937	0.0179	0.1223	0.2409	1.8220	0.5057	0.7642	1.0917	0.0669	0.0139	0.0897	0.1370	2.0065	0.4990	1.0007	0.9948	0.0520	0.0127	0.0701	0.0772
500	2	0.5	2	1	1.6808	0.5155	1.0873	1.1661	0.0924	0.0199	0.2350	0.1905	1.7504	0.5242	1.4789	1.1815	0.0716	0.0158	0.1707	0.1010	2.0034	0.4996	2.0033	1.0016	0.0545	0.0128	0.1341	0.0781
500	2	0.5	0.5	2	1.8592	0.4739	0.3955	1.5547	0.1126	0.0188	0.0844	0.3595	1.9128	0.4903	0.4524	1.7880	0.0874	0.0155	0.0554	0.2363	2.0048	0.4994	0.5008	1.9939	0.0516	0.0131	0.0380	0.1616
500	2	0.5	1	2	1.9185	0.4786	0.7807	1.7604	0.0848	0.0140	0.0997	0.2335	1.9667	0.4936	0.9278	1.9288	0.0627	0.0120	0.0725	0.1571	2.0068	0.4988	0.9953	1.9932	0.0526	0.0128	0.0692	0.1531
500	2	0.5	2	2	1.9072	0.4901	1.6060	1.9233	0.0778	0.0135	0.1539	0.1609	1.9744	0.4975	1.8893	1.9881	0.0584	0.0124	0.1276	0.1459	2.0052	0.5001	2.0020	2.0038	0.0565	0.0131	0.1329	0.1409

4.2. Type-II Censoring

In case of Type-II censoring, we considered different censoring rates 10%, 20% and 30% with different values of n.

4.2.1. Exponential distribution

We conducted a simulation study using a baseline exponential distribution with a rate parameter $\theta = 2$, and $n = 30, 100, 500$. This study was repeated for two subsystem configurations: "1-out-of-2" and "2-out-of-4". For the "1-out-of-2" subsystem, we considered $\beta = 0.2, 0.5, 1, 1.5, 2$. For the "2-out-of-4" subsystem, we used $\beta_1, \beta_2 = 0.5, 1, 2$. We calculated the average parameter estimates for θ and $\beta_j, j = 1, 2$, along with their standard errors (SE)based on 1000 repetitions, for the "1-out-of-2" case and the "2-out-of-4" case with $q = 0.1 * n, 0.2 * n, 0.3 * n$, using the Type-II censoring approach.

The results are presented in Tables 5 and 6.

From Table 5, we observe that as the q increases (resulting in more censored observations), both the bias and the variability of the estimates for θ increase. The estimates for the load sharing parameter β exhibit similar behavior. Additionally, as β increases, the bias also increases. However, as the number of parallel subsystems n increases, the bias and variability in the estimates decrease.

Table 6 shows that an increase in q leads to increased bias and variability in the estimates for both θ and β. While the bias and variability of θ decrease as n increases, the estimates for β are not significantly affected by different values of $n = 30, 100, 500$.

Table 5 Parameter estimates with varying number of failure ,baseline parameter $\theta = 2$ and different values of n for 1-out-of-2 subsystem with Exponential baseline distribution.

		q=0.1*n				q=0.2*n				q=0.3*n			
n	β	$\hat{\theta}$	$\hat{\beta}$	SE of θ	SE of β	$\hat{\theta}$	$\hat{\beta}$	SE of θ	SE of β	$\hat{\theta}$	$\hat{\beta}$	SE of θ	SE of β
30	0.2	2.0919	0.2074	0.4256	0.0586	2.1721	0.2016	0.4724	0.0594	2.2099	0.1993	0.4849	0.0661
30	0.5	2.1458	0.5005	0.4203	0.1420	2.2530	0.4906	0.4607	0.1728	2.4375	0.4648	0.5559	0.1944
30	1	2.2775	0.9426	0.4503	0.2775	2.5135	0.8584	0.5022	0.3015	2.7847	0.7943	0.6238	0.3391
30	1.5	2.3784	1.3602	0.4835	0.4282	2.6698	1.1977	0.5354	0.4297	3.0801	1.0342	0.6745	0.4742
30	2	2.4549	1.7100	0.4510	0.5156	2.8799	1.4404	0.5948	0.5491	3.2685	1.2408	0.7009	0.5656
100	0.2	2.0584	0.1977	0.2186	0.0299	2.0991	0.1949	0.2427	0.0321	2.1299	0.1948	0.2666	0.0371
100	0.5	2.0944	0.4874	0.2134	0.0757	2.1934	0.4743	0.2432	0.0868	2.3475	0.4445	0.2637	0.1113
100	1	2.2116	0.9211	0.2274	0.1590	2.4319	0.8471	0.2709	0.2070	2.7033	0.7601	0.3198	0.2333
100	1.5	2.4391	1.6300	0.2471	0.3285	2.6431	1.1363	0.2947	0.2944	2.9906	1.0134	0.3436	0.3460
100	2	2.4419	1.6450	0.2580	0.3316	2.8019	1.3923	0.3079	0.3708	3.2186	1.1913	0.3772	0.4109
500	0.2	2.0290	0.1979	0.0969	0.0133	2.0659	0.1950	0.1010	0.0151	2.1026	0.1917	0.1138	0.0176
500	0.5	2.0805	0.4833	0.0975	0.0388	2.1863	0.4604	0.1087	0.0496	2.3161	0.4401	0.1192	0.0761
500	1	2.2056	0.9084	0.1033	0.0985	2.4271	0.8220	0.1170	0.1448	2.6769	0.7534	0.1369	0.1945
500	1.5	2.3229	1.2740	0.1089	0.1621	2.6236	1.1116	0.1263	0.2306	2.9583	0.9882	0.1462	0.2780
500	2	2.4179	1.6195	0.1108	0.2259	2.7879	1.3712	0.1386	0.3044	3.1871	1.1728	0.1615	0.3505

Table 6 Parameter estimates with varying number of failure ,baseline parameter $\theta = 2$ and different values of n for 2-out-of-4 subsystem with Exponential baseline distribution.

			q=0.1*n						q=0.2*n						q=0.3*n					
n	β_1	β_2	$\hat{\theta}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of θ	SE of β_1	SE of β_2	$\hat{\theta}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of θ	SE of β_1	SE of β_2	$\hat{\theta}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of θ	SE of β_1	SE of β_2
30	0.5	0.5	2.1290	0.6657	0.6657	0.4313	0.1826	0.1826	2.1872	0.8092	0.8092	0.4711	0.2488	0.2488	2.2648	0.9723	0.9723	0.5074	0.3070	0.3070
30	0.5	1	2.1293	0.6662	1.3324	0.4153	0.1855	0.3710	2.2181	0.7918	1.5836	0.4401	0.2348	0.4696	2.3758	0.9038	1.8077	0.5505	0.2812	0.5624
30	0.5	2	2.1718	0.6439	2.5755	0.4416	0.1743	0.6971	2.2750	0.7623	3.0493	0.4578	0.2178	0.8713	2.4532	0.8576	3.4304	0.5891	0.2674	1.0694
100	0.5	0.5	2.0499	0.6660	0.6660	0.2169	0.1000	0.1000	2.1233	0.7968	0.7968	0.2417	0.1310	0.1310	2.1846	0.9461	0.9461	0.2639	0.1589	0.1589
100	0.5	1	2.0907	0.6481	1.2961	0.2290	0.0981	0.1962	2.1761	0.7732	1.5464	0.2436	0.1233	0.2466	2.2858	0.8943	1.7885	0.2787	0.1523	0.3047
100	0.5	2	2.1264	0.6407	2.5630	0.2251	0.0963	0.3853	2.2257	0.7482	2.9927	0.2404	0.1188	0.4754	2.3812	0.8496	3.3986	0.2852	0.1418	0.5670
500	0.5	0.5	2.0478	0.6572	0.6572	0.0961	0.0447	0.0447	2.0992	0.7965	0.7965	0.1045	0.0560	0.0560	2.1697	0.9447	0.9447	0.1102	0.0702	0.0702
500	0.5	1	2.0693	0.6483	1.2967	0.1015	0.0444	0.0887	2.1530	0.7693	1.5386	0.1115	0.0549	0.1098	2.2627	0.8941	1.7883	0.1174	0.0664	0.1329
500	0.5	2	2.0943	0.6369	2.5477	0.0971	0.0413	0.1653	2.2084	0.7444	2.9776	0.1091	0.0524	0.2098	2.3481	0.8450	3.3800	0.1237	0.0643	0.2572
30	1	0.5	2.1345	1.3292	0.6646	0.4298	0.3633	0.1817	2.2073	1.5886	0.7943	0.4771	0.4595	0.2298	2.3655	1.8552	0.9276	0.5401	0.5951	0.2976
30	1	1	2.1643	1.2968	1.2968	0.4477	0.3591	0.3591	2.3275	1.4873	1.4872	0.4880	0.4190	0.4190	2.4929	1.6695	1.6695	0.5673	0.4999	0.4999
30	1	2	2.2393	1.2245	2.4491	0.4333	0.3250	0.6501	2.4404	1.3766	2.7533	0.5049	0.3759	0.7517	2.7089	1.4722	2.9445	0.5975	0.4305	0.8609
100	1	0.5	2.0815	1.3066	0.6533	0.2274	0.2016	0.1008	2.1396	1.5772	0.7886	0.2441	0.2574	0.1287	2.2519	1.8356	0.9178	0.2628	0.3073	0.1537
100	1	1	2.1229	1.2768	1.2768	0.2333	0.1973	0.1973	2.2462	1.4706	1.4706	0.2503	0.2285	0.2285	2.4138	1.6553	1.6553	0.2849	0.2746	0.2746
100	1	2	2.1987	1.2102	2.4203	0.2311	0.1804	0.3608	2.3639	1.3615	2.7229	0.2651	0.2058	0.4116	2.5969	1.4694	2.9388	0.2984	0.2340	0.4681
500	1	0.5	2.0575	1.3053	0.6527	0.0998	0.0910	0.0455	2.1309	1.5556	0.7778	0.1037	0.1082	0.0541	2.2258	1.8296	0.9148	0.1149	0.1306	0.0653
500	1	1	2.1101	1.2627	1.2627	0.0999	0.0837	0.0837	2.2323	1.4655	1.4655	0.1086	0.0995	0.0995	2.3927	1.6419	1.6419	0.1261	0.1207	0.1206
500	1	2	2.1648	1.2147	2.4293	0.0972	0.0804	0.1608	2.3637	1.3477	2.6953	0.1204	0.0925	0.1850	2.5890	1.4570	2.9139	0.1356	0.0995	0.1990
30	2	0.5	2.1341	2.6608	0.6652	0.4238	0.7684	0.1921	2.2314	3.1633	0.7908	0.4697	0.9408	0.2352	2.3776	3.6272	0.9068	0.5417	1.1483	0.2871
30	2	1	2.2219	2.5298	1.2649	0.4404	0.6836	0.3418	2.3823	2.8560	1.4280	0.5029	0.8070	0.4035	2.5985	3.1751	1.5875	0.5926	0.9807	0.4903
30	2	2	2.3156	2.3269	2.3269	0.4652	0.6298	0.6298	2.5644	2.5011	2.5012	0.5255	0.6988	0.6987	2.9012	2.6298	2.6299	0.6488	0.7697	0.7697
100	2	0.5	2.0847	2.5972	0.6493	0.2150	0.3821	0.0955	2.1825	3.1000	0.7750	0.2369	0.4831	0.1208	2.2820	3.5964	0.8991	0.2721	0.6037	0.1509
100	2	1	2.1553	2.4956	1.2478	0.2239	0.3662	0.1831	2.3304	2.8070	1.4035	0.2635	0.4350	0.2175	2.5330	3.0839	1.5420	0.2902	0.5012	0.2506
100	2	2	2.2497	2.3202	2.3202	0.2273	0.3294	0.3294	2.5270	2.4401	2.4401	0.2838	0.3703	0.3703	2.8337	2.5435	2.5435	0.3170	0.3884	0.3884
500	2	0.5	2.0636	2.6043	0.6511	0.0983	0.1800	0.0450	2.1485	3.0877	0.7719	0.1063	0.2178	0.0545	2.2601	3.5696	0.8924	0.1174	0.2632	0.0658
500	2	1	2.1335	2.4841	1.2421	0.0969	0.1588	0.0794	2.3033	2.8027	1.4014	0.1101	0.1908	0.0954	2.5069	3.0684	1.5342	0.1294	0.2181	0.1091
500	2	2	2.1413	2.4703	1.2352	0.1009	0.1652	0.0826	2.5086	2.4281	2.4282	0.1228	0.1604	0.1604	2.8000	2.5493	2.5493	0.1453	0.1746	0.1746

4.2.2. Weibull distribution

We have taken baseline distribution as Weibull (a, b). To asses the performances of estimates for two different configurations composed by "1-out-of-2" and "2-out-of-4" subsystems, we have conducted a simulation study.

For the "1-out-of-2" subsystem under a Type-II censoring approach, we considered $n = 30, 100, 500$ with various parameter combinations: $a, b = 0.5, 2$ and $\beta = 0.5, 1, 2$ at different values of $q = 0.1 * n, 0.2 * n, 0.3 * n$. The average parameter estimates for a, b, and β, along with their standard errors (SE) based on 1000 repetitions, are presented in Table 7.

For the "2-out-of-4" subsystem under the Type-II censoring approach, we used $n = 30, 100, 500$ and parameter combinations: $a, b = 0.5, 2$ with $\beta_1, \beta_2 = 0.5, 1, 2$ at $q = 0.1 * n, 0.2 * n, 0.3 * n$. The average parameter estimates for a, b, and β_j (for $j = 1, 2$), along with their standard errors (SE) based on 1000 repetitions, are shown in Table 8.

From Table 7, we observe that as the censoring time increases, the bias in the estimates of the shape parameter a decreases. Similarly, as the sample size n increases, the bias in the estimates of a decreases. However, the bias in the estimates of a increases with higher values of a. Additionally, the termination/censoring time t_0 and the shape parameter a affect the scale parameter b; the bias in b decreases with increasing t_0 and a. However, the bias in b increases as b increases. For the load-sharing parameter β, its bias decreases with increasing t_0 and b, but increases with higher values of a and β.

Similar conclusions can be drawn from Table 8 for the "2-out-of-4" subsystem under the Type-II censoring approach.

Table 7 Parameter estimates with varying number of failure with different baseline parameters and different values of n for 1-out-of-2 subsystem with Weibull baseline distribution.

				q=0.1*n						q=0.2*n						q=0.3*n					
n	a	b	β_1	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	SE of a	SE of b	SE of β_1	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	SE of a	SE of b	SE of β_1	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	SE of a	SE of b	SE of β_1
30	0.5	2	0.5	0.5273	1.8712	0.4695	0.0705	0.7413	0.1718	0.5347	1.7227	0.4414	0.0759	0.7371	0.1800	0.5503	1.5198	0.4131	0.0881	0.6948	0.2011
30	0.5	2	1	0.5361	1.6662	0.8834	0.0700	0.6246	0.3121	0.5465	1.3723	0.7924	0.0821	0.5553	0.3299	0.5619	1.1275	0.7180	0.1002	0.4936	0.3643
30	0.5	2	2	0.5495	1.3684	1.5516	0.0789	0.5213	0.5319	0.5632	1.0295	1.3032	0.0866	0.3897	0.5324	0.5738	0.7746	1.0830	0.0982	0.3194	0.5281
100	0.5	2	0.5	0.5101	1.8744	0.4803	0.0370	0.4095	0.0989	0.5165	1.6670	0.4457	0.0400	0.3573	0.1026	0.5204	1.4994	0.4271	0.0475	0.3566	0.1320
100	0.5	2	1	0.5180	1.6468	0.8948	0.0377	0.3311	0.1781	0.5266	1.3463	0.7833	0.0420	0.2865	0.1972	0.5376	1.1024	0.6995	0.0494	0.2593	0.2293
100	0.5	2	2	0.5344	1.3690	1.5460	0.0401	0.2841	0.3144	0.5424	1.0092	1.2928	0.0465	0.2182	0.3463	0.5559	0.7611	1.0631	0.0541	0.1748	0.3640
500	0.5	2	0.5	0.5040	1.8541	0.4784	0.0171	0.1770	0.0506	0.5072	1.6822	0.4564	0.0200	0.1728	0.0675	0.5107	1.4998	0.4313	0.0246	0.1562	0.0911
500	0.5	2	1	0.5131	1.6276	0.8783	0.0172	0.1509	0.1020	0.5212	1.3415	0.7868	0.0230	0.1300	0.1453	0.5288	1.0912	0.7069	0.0284	0.1154	0.1814
500	0.5	2	2	0.5254	1.3356	1.5211	0.0225	0.1213	0.1922	0.5367	0.9907	1.2701	0.0283	0.0996	0.2533	0.5446	0.7492	1.0514	0.0315	0.0767	0.2944
30	2	0.5	0.5	2.1131	0.4856	0.4684	0.2758	0.0472	0.1758	2.1468	0.4732	0.4425	0.3138	0.0480	0.1805	2.1738	0.4563	0.4176	0.3392	0.0501	0.1949
30	2	0.5	1	2.1453	0.4700	0.8747	0.3000	0.0455	0.3116	2.1661	0.4483	0.8024	0.3190	0.0427	0.3250	2.2413	0.4254	0.7016	0.3837	0.0447	0.3374
30	2	0.5	2	2.1917	0.4520	1.5789	0.3086	0.0423	0.5438	2.2519	0.4211	1.3337	0.3494	0.0413	0.5541	2.3179	0.3902	1.1059	0.4237	0.0404	0.5531
100	2	0.5	0.5	2.0400	0.4881	0.4714	0.1498	0.0249	0.0952	2.0550	0.4774	0.4547	0.1634	0.0269	0.1110	2.0855	0.4621	0.4230	0.1789	0.0270	0.1266
100	2	0.5	1	2.0740	0.4742	0.8808	0.1492	0.0246	0.1752	2.1074	0.4512	0.7837	0.1698	0.0246	0.2040	2.1541	0.4281	0.7025	0.1993	0.0241	0.2358
100	2	0.5	2	2.1239	0.4516	1.5270	0.1606	0.0234	0.3131	2.1688	0.4194	1.2911	0.1838	0.0216	0.3449	2.2061	0.3916	1.0979	0.2232	0.0224	0.3678
500	2	0.5	0.5	2.0150	0.4902	0.4794	0.0697	0.0116	0.0495	2.0303	0.4783	0.4548	0.0790	0.0117	0.0665	2.0532	0.4627	0.4233	0.1007	0.0124	0.0903
500	2	0.5	1	2.0518	0.4751	0.8817	0.0712	0.0111	0.1022	2.0894	0.4509	0.7839	0.0910	0.0109	0.1455	2.1206	0.4288	0.7014	0.1159	0.0114	0.1837
500	2	0.5	2	2.1051	0.4520	1.5221	0.0850	0.0103	0.1996	2.1463	0.4193	1.2767	0.1103	0.0104	0.2544	2.1884	0.3907	1.0744	0.1327	0.0104	0.3016

Table 8 Parameter estimates with varying number of failure with different baseline parameters and different values of n for 2-out-of-4 subsystem with Weibull baseline distribution.

					q=0.1*n								q=0.2*n								q=0.3*n							
n	a	b	β_1	β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of a	SE of b	SE of β_1	SE of β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of a	SE of b	SE of β_1	SE of β_2	$\hat{a}$	$\hat{b}$	$\hat{\beta}_1$	$\hat{\beta}_2$	SE of a	SE of b	SE of β_1	SE of β_2
30	0.5	2	0.5	0.5	0.5233	1.8872	0.4482	0.4750	0.0681	0.8998	0.1580	0.2084	0.5283	1.7593	0.4182	0.4606	0.0779	0.8878	0.1693	0.2573	0.5356	1.6597	0.3911	0.4575	0.0927	0.9349	0.1805	0.3025
30	0.5	2	1	0.5	0.5210	1.8906	0.8968	0.4801	0.0676	0.8953	0.2966	0.2050	0.5259	1.7355	0.7858	0.4772	0.0827	0.9537	0.3091	0.2632	0.5353	1.6020	0.7050	0.4709	0.1003	0.9908	0.3211	0.3131
30	0.5	2	2	0.5	0.5192	1.8821	1.7840	0.4931	0.0683	0.8665	0.5448	0.1948	0.5246	1.7372	1.5523	0.4710	0.0828	0.9617	0.5627	0.2392	0.5358	1.5962	1.3649	0.4710	0.0988	0.9340	0.5577	0.2969
30	0.5	2	0.5	1	0.5235	1.8225	0.4640	0.8937	0.0667	0.8853	0.1690	0.3759	0.5339	1.6275	0.4368	0.8088	0.0814	0.8489	0.1848	0.4076	0.5390	1.4553	0.4035	0.7620	0.0950	0.8202	0.2009	0.4565
30	0.5	2	1	1	0.5276	1.6797	0.8573	0.9105	0.0702	0.7603	0.2791	0.3771	0.5330	1.4830	0.7864	0.8423	0.0822	0.7747	0.3245	0.4250	0.5420	1.2530	0.6771	0.7681	0.0963	0.6991	0.3055	0.4731
30	0.5	2	2	1	0.5271	1.6814	1.6839	0.9006	0.0720	0.7882	0.5137	0.3710	0.5325	1.4257	1.4864	0.8512	0.0826	0.7490	0.5644	0.4293	0.5422	1.1815	1.2376	0.7776	0.1002	0.6444	0.5306	0.4560
30	0.5	2	0.5	2	0.5185	1.7883	0.4747	1.7404	0.0653	0.7985	0.1629	0.6724	0.5151	1.6128	0.4532	1.5576	0.0724	0.7712	0.1829	0.6781	0.5299	1.3735	0.4170	1.3509	0.0865	0.7370	0.1943	0.7166
30	0.5	2	1	2	0.5303	1.5246	0.8475	1.6190	0.0676	0.7117	0.2914	0.6232	0.5358	1.2489	0.7608	1.4082	0.0801	0.5934	0.3028	0.6463	0.5428	1.0311	0.6659	1.2447	0.0902	0.5303	0.3013	0.6658
30	0.5	2	2	2	0.5326	1.4006	1.5708	1.5962	0.0706	0.6165	0.4941	0.5969	0.5378	1.1007	1.3136	1.4110	0.0823	0.5272	0.4758	0.6258	0.5502	0.8709	1.1272	1.1827	0.0999	0.4710	0.5290	0.6314
100	0.5	2	0.5	0.5	0.5044	1.8943	0.4646	0.4848	0.0363	0.4813	0.0907	0.1406	0.5069	1.7774	0.4226	0.4778	0.0498	0.5244	0.1105	0.1926	0.5132	1.6125	0.3901	0.4524	0.0617	0.5173	0.1181	0.2217
100	0.5	2	1	0.5	0.5078	1.8459	0.8904	0.4782	0.0412	0.4835	0.1708	0.1230	0.5095	1.7360	0.7955	0.4778	0.0523	0.5592	0.1923	0.1876	0.5179	1.5530	0.6990	0.4530	0.0673	0.5399	0.2154	0.2225
100	0.5	2	2	0.5	0.5043	1.8968	1.7568	0.4921	0.0393	0.5081	0.3171	0.1216	0.5068	1.7431	1.5644	0.4775	0.0535	0.5507	0.3551	0.1642	0.5093	1.6034	1.3518	0.4711	0.0690	0.6143	0.4270	0.2165
100	0.5	2	0.5	1	0.5051	1.8204	0.4681	0.9143	0.0353	0.4507	0.0906	0.2387	0.5061	1.6392	0.4335	0.8602	0.0440	0.4421	0.1139	0.2940	0.5108	1.4395	0.4041	0.7910	0.0547	0.4257	0.1256	0.3617
100	0.5	2	1	1	0.5108	1.6869	0.8700	0.9014	0.0399	0.4332	0.1754	0.2391	0.5156	1.4458	0.7664	0.8295	0.0501	0.4049	0.1912	0.3025	0.5177	1.2884	0.6957	0.7861	0.0629	0.4120	0.2164	0.3613
100	0.5	2	2	1	0.5095	1.6543	1.6637	0.9127	0.0423	0.4421	0.3111	0.2299	0.5148	1.3889	1.4266	0.8316	0.0512	0.4007	0.3393	0.2874	0.5217	1.1630	1.2359	0.7626	0.0682	0.3948	0.3770	0.3451
100	0.5	2	0.5	2	0.5013	1.7979	0.4784	1.7589	0.0333	0.4395	0.0947	0.3942	0.5013	1.5648	0.4521	1.5408	0.0389	0.4084	0.1156	0.4667	0.5057	1.3814	0.4287	1.3532	0.0494	0.3873	0.1381	0.5323
100	0.5	2	1	2	0.5120	1.5300	0.8558	1.6580	0.0379	0.3642	0.1745	0.3996	0.5152	1.2541	0.7454	1.4542	0.0447	0.3289	0.1957	0.4609	0.5196	1.0500	0.6679	1.2669	0.0582	0.3093	0.2198	0.5158
100	0.5	2	2	2	0.5155	1.3808	1.5560	1.5912	0.0412	0.3595	0.3064	0.3886	0.5223	1.0762	1.3008	1.3820	0.0507	0.2976	0.3426	0.4633	0.5252	0.8645	1.1161	1.2072	0.0658	0.2675	0.3503	0.5292
500	0.5	2	0.5	0.5	0.4993	1.9004	0.4595	0.4903	0.0245	0.2504	0.0541	0.1050	0.5013	1.7729	0.4238	0.4779	0.0366	0.3089	0.0744	0.1625	0.5032	1.6426	0.3931	0.4654	0.0486	0.3531	0.0936	0.2095
500	0.5	2	1	0.5	0.4995	1.8886	0.8912	0.4929	0.0259	0.2746	0.1003	0.0963	0.5020	1.7350	0.7882	0.4792	0.0396	0.3348	0.1436	0.1522	0.5035	1.5967	0.7106	0.4738	0.0536	0.3918	0.1767	0.2072
500	0.5	2	2	0.5	0.4997	1.8905	1.7803	0.4917	0.0245	0.2655	0.1928	0.0799	0.5040	1.7164	1.5428	0.4673	0.0401	0.3460	0.2790	0.1218	0.4993	1.6114	1.3492	0.4851	0.0566	0.4677	0.3289	0.2089
500	0.5	2	0.5	1	0.4996	1.8200	0.4714	0.9148	0.0196	0.2164	0.0572	0.1757	0.5003	1.6370	0.4399	0.8511	0.0289	0.2363	0.0808	0.2488	0.5038	1.4687	0.4125	0.7870	0.0426	0.2715	0.1002	0.3151
500	0.5	2	1	1	0.5035	1.6947	0.8666	0.9128	0.0255	0.2285	0.1087	0.1834	0.5085	1.4536	0.7710	0.8364	0.0367	0.2395	0.1457	0.2573	0.5109	1.2579	0.6897	0.7916	0.0494	0.2550	0.1816	0.3243
500	0.5	2	2	1	0.5049	1.6592	1.6741	0.9092	0.0279	0.2394	0.1934	0.1699	0.5080	1.3983	1.4252	0.8388	0.0422	0.2771	0.2678	0.2583	0.5119	1.1720	1.2268	0.7787	0.0563	0.2785	0.3132	0.3177
500	0.5	2	0.5	2	0.4954	1.8042	0.4776	1.7834	0.0163	0.1997	0.0566	0.2440	0.4944	1.5855	0.4521	1.5616	0.0215	0.2019	0.0827	0.3621	0.4979	1.3750	0.4235	1.3933	0.0325	0.2100	0.1055	0.4499
500	0.5	2	1	2	0.5066	1.5295	0.8565	1.6444	0.0206	0.1785	0.1087	0.2903	0.5113	1.2412	0.7511	1.4279	0.0312	0.1775	0.1519	0.4075	0.5106	1.0310	0.6674	1.2867	0.0397	0.1753	0.1824	0.4578
500	0.5	2	2	2	0.5108	1.3757	1.5558	1.5867	0.0236	0.1735	0.1899	0.2796	0.5160	1.0742	1.3077	1.3715	0.0377	0.1745	0.2568	0.3955	0.5196	0.8633	1.1178	1.2150	0.0532	0.1764	0.2884	0.4678
30	2	0.5	0.5	0.5	2.0852	0.4830	0.4559	0.4804	0.2729	0.0553	0.1570	0.2155	2.1113	0.4732	0.4180	0.4643	0.3207	0.0586	0.1749	0.2618	2.1456	0.4596	0.3776	0.4394	0.3679	0.0594	0.1801	0.2825
30	2	0.5	1	0.5	2.1119	0.4789	0.8797	0.4692	0.2808	0.0547	0.2999	0.1981	2.0959	0.4710	0.7857	0.4736	0.3194	0.0585	0.3060	0.2436	2.1429	0.4565	0.6938	0.4558	0.3755	0.0634	0.3247	0.2860
30	2	0.5	2	0.5	2.0925	0.4835	1.7741	0.4814	0.2764	0.0544	0.5584	0.1935	2.0975	0.4722	1.5697	0.4667	0.3280	0.0594	0.5691	0.2169	2.1492	0.4561	1.3370	0.4525	0.3946	0.0635	0.5691	0.2834
30	2	0.5	0.5	1	2.0877	0.4787	0.4682	0.9141	0.2666	0.0542	0.1682	0.3983	2.0992	0.4659	0.4367	0.8248	0.3135	0.0547	0.1821	0.4082	2.1341	0.4513	0.4024	0.7855	0.3647	0.0591	0.1957	0.4485
30	2	0.5	1	1	2.1034	0.4724	0.8746	0.9135	0.2650	0.0520	0.2969	0.3656	2.1347	0.4531	0.7789	0.8260	0.3352	0.0529	0.3112	0.3845	2.1548	0.4375	0.6832	0.7887	0.3763	0.0559	0.3090	0.4485
30	2	0.5	2	1	2.1231	0.4662	1.6767	0.8846	0.2913	0.0530	0.5196	0.3526	2.1216	0.4505	1.4425	0.8710	0.3393	0.0542	0.5397	0.4211	2.1717	0.4262	1.2356	0.7775	0.4061	0.0583	0.5185	0.4683
30	2	0.5	0.5	2	2.0788	0.4762	0.4702	1.7264	0.2710	0.0544	0.1663	0.6724	2.1018	0.4606	0.4448	1.4975	0.2986	0.0541	0.1794	0.6734	2.1071	0.4430	0.4126	1.3171	0.3475	0.0572	0.1897	0.6880
30	2	0.5	1	2	2.1210	0.4610	0.8703	1.6511	0.2815	0.0507	0.2961	0.6267	2.1462	0.4362	0.7711	1.4246	0.3208	0.0483	0.3144	0.6491	2.1837	0.4125	0.6729	1.2292	0.3818	0.0521	0.3128	0.6738
30	2	0.5	2	2	2.1431	0.4478	1.5736	1.5765	0.2781	0.0483	0.5067	0.5950	2.1808	0.4209	1.3298	1.3698	0.3522	0.0504	0.5126	0.6392	2.1859	0.3980	1.1399	1.2130	0.4216	0.0487	0.5258	0.6336
100	2	0.5	0.5	0.5	2.0220	0.4896	0.4589	0.4864	0.1619	0.0325	0.0951	0.1463	2.0277	0.4830	0.4313	0.4764	0.2054	0.0367	0.1028	0.2005	2.0488	0.4722	0.3944	0.4567	0.2416	0.0402	0.1179	0.2350
100	2	0.5	1	0.5	2.0193	0.4908	0.8936	0.4928	0.1598	0.0327	0.1683	0.1380	2.0216	0.4812	0.7888	0.4849	0.2115	0.0355	0.1906	0.1793	2.0522	0.4674	0.7092	0.4732	0.2627	0.0398	0.2148	0.2412
100	2	0.5	2	0.5	2.0327	0.4867	1.7592	0.4785	0.1617	0.0318	0.3213	0.1146	2.0298	0.4776	1.5458	0.4703	0.2082	0.0364	0.3576	0.1602	2.0398	0.4649	1.3676	0.4656	0.2709	0.0427	0.4041	0.2115
100	2	0.5	0.5	1	2.0155	0.4863	0.4716	0.9130	0.1450	0.0304	0.0924	0.2400	2.0281	0.4723	0.4394	0.8452	0.1789	0.0326	0.1146	0.3025	2.0504	0.4584	0.4128	0.7778	0.2230	0.0341	0.1292	0.3585
100	2	0.5	1	1	2.0398	0.4777	0.8706	0.9148	0.1600	0.0304	0.1729	0.2514	2.0618	0.4566	0.7641	0.8209	0.2051	0.0322	0.1973	0.2911	2.0874	0.4381	0.6824	0.7666	0.2574	0.0352	0.2170	0.3665
100	2	0.5	2	1	2.0426	0.4742	1.6759	0.9011	0.1636	0.0301	0.3028	0.2255	2.0643	0.4523	1.4342	0.8343	0.2236	0.0332	0.3494	0.3051	2.0861	0.4319	1.2253	0.7772	0.2718	0.0355	0.3708	0.3681
100	2	0.5	0.5	2	2.0060	0.4841	0.4777	1.7466	0.1365	0.0289	0.0938	0.4096	2.0052	0.4688	0.4442	1.5551	0.1627	0.0301	0.1114	0.4620	2.0189	0.4514	0.4161	1.3874	0.1933	0.0322	0.1373	0.5139
100	2	0.5	1	2	2.0546	0.4656	0.8514	1.6304	0.1515	0.0281	0.1719	0.3940	2.0668	0.4421	0.7503	1.4366	0.1882	0.0294	0.2019	0.4736	2.0792	0.4197	0.6743	1.2625	0.2284	0.0313	0.2350	0.5255
100	2	0.5	2	2	2.0645	0.4552	1.5795	1.5930	0.1541	0.0277	0.3149	0.3832	2.0810	0.4260	1.3089	1.3924	0.2070	0.0287	0.3315	0.4706	2.1051	0.4023	1.1064	1.2164	0.2581	0.0300	0.3436	0.5078
500	2	0.5	0.5	0.5	2.0012	0.4924	0.4620	0.4862	0.1003	0.0166	0.0533	0.1076	2.0810	0.4260	1.3089	1.3924	0.2070	0.0287	0.3315	0.4706	2.0035	0.4764	0.3918	0.4803	0.1897	0.0249	0.0966	0.2127
500	2	0.5	1	0.5	1.9978	0.4911	0.8831	0.4911	0.1048	0.0169	0.0996	0.0951	2.0034	0.4815	0.7922	0.4822	0.1582	0.0233	0.1438	0.1561	2.0191	0.4696	0.7005	0.4745	0.2160	0.0287	0.1752	0.2170
500	2	0.5	2	0.5	2.0051	0.4904	1.7679	0.4853	0.0976	0.0158	0.1984	0.0749	2.0068	0.4811	1.5556	0.4768	0.1668	0.0244	0.2670	0.1331	2.0103	0.4691	1.3389	0.4712	0.2299	0.0324	0.3263	0.1959
500	2	0.5	0.5	1	1.9955	0.4884	0.4704	0.9229	0.0785	0.0147	0.0570	0.1754	2.0055	0.4743	0.4469	0.8374	0.1254	0.0178	0.0816	0.2519	2.0081	0.4615	0.4118	0.7913	0.1593	0.0209	0.1035	0.3052
500	2	0.5	1	1	2.0110	0.4801	0.8671	0.9263	0.1004	0.0159	0.1094	0.1822	2.0322	0.4611	0.7729	0.8446	0.1496	0.0194	0.1473	0.2654	2.0486	0.4428	0.6720	0.7736	0.1945	0.0229	0.1842	0.3192
500	2	0.5	2	1	2.0082	0.4770	1.6692	0.9182	0.1070	0.0173	0.1976	0.1689	2.0433	0.4535	1.4144	0.8248	0.1661	0.0207	0.2641	0.2443	2.0499	0.4346	1.2145	0.7732	0.2306	0.0262	0.3011	0.3192
500	2	0.5	0.5	2	1.9814	0.4866	0.4808	1.7601	0.0635	0.0135	0.0604	0.2435	1.9809	0.4706	0.4473	1.5639	0.0889	0.0153	0.0836	0.3558	1.9913	0.4532	0.4253	1.3603	0.1288	0.0177	0.1112	0.4476
500	2	0.5	1	2	2.0245	0.4677	0.8555	1.6487	0.0785	0.0134	0.1091	0.2791	2.0404	0.4439	0.7551	1.4464	0.1194	0.0156	0.1509	0.3898	2.0460	0.4223	0.6584	1.2713	0.1547	0.0178	0.1832	0.4384
500	2	0.5	2	2	2.0462	0.4552	1.5601	1.5864	0.0967	0.0140	0.1987	0.2846	2.0629	0.4270	1.3085	1.3836	0.1463	0.0168	0.2656	0.3859	2.0737	0.4030	1.1144	1.2087	0.2013	0.0198	0.2814	0.4630

5. Data Analysis

The suggested parallel load-sharing configuration with k-out-of-m subsystems finds practical applications in various real-life scenarios, a concept we can exemplify through an illustration.

Type-I censoring scheme

The data was taken from Reliability Edge Home [23] and we call it as 'Motor Data'. Let's consider a life test conducted on 18 parallel subsystems, each consisting of two continuously operating motors. The test recorded the failure times of both motors for each subsystem, along with their identification labels. These 18 parallel subsystems can be viewed as 1-out-of-2 subsystems, connected in either parallel or series to create a unified configuration. This integrated system can be depicted in parallel, with all 18 subsystems operating independently, or in series, necessitating the operation of all 18 subsystems for the entire system to function as a series connection. Our investigation specifically emphasizes the parallel configuration.

Under Type-I censoring scheme we consider three termination points as $t_0 = 300, 275$ and 220. To assess the reliability of the scheme, assuming they are organized in parallel. We have assume that the component baseline distribution as the Exponential and Weibull. The parameter estimates obtained by classical approach. along with Akaike Information Criteria (AIC) ([24]) and Bayesian Information Criteria (BIC) ([25]) for Exponential distribution and Weibull distribution are given in the following Table 9.

Table 9 Parameter estimates with AIC and BIC values for Motor Data.

	t_0=300	
	Exponential	Weibull
Parameter Estimates	$\hat{\theta}$=0.0029 , $\hat{\beta_1}$=7.2026	$\hat{a}$=2.8935 , $\hat{b}$=241.8706 , $\hat{\beta_1}$= 2.2729
AIC Values	379.1674	358.5163
BIC Values	383.8385	361.1874
	t_0=275	
Parameter Estimates	$\hat{\theta}$=0.0032 , $\hat{\beta_1}$=5.2158	$\hat{a}$=2.8089 , $\hat{b}$= 219.4505, $\hat{\beta_1}$=1.4993
AIC Values	316.4922	301.6024
BIC Values	321.1633	304.2735
	t_0=220	
Parameter Estimates	$\hat{\theta}$=0.0034, $\hat{\beta_1}$= 6.2496	$\hat{a}$=3.1675 , $\hat{b}$=203.9398 , $\hat{\beta_1}$= 1.8618
AIC Values	264.3963	247.7652
BIC Values	269.0674	250.4363

Table 9 reveals that a lower AIC or BIC value, indicative of an Weibull distribution, signifies a better fit. Therefore, we opt for the Weibull model based on the AIC and BIC values presented in the table above. And as censoring increases the model fits better. The associated reliability of the model corresponding to the above said censoring schemes are $0.5738, 0.8431$and 0.9999 respectively.

Type-II censoring scheme

Under Type-II censoring scheme we consider three failure points as $q = 2, 4$ and 5. To assess the reliability of the scheme, assuming they are organized in parallel. We have assume that the component baseline distribution as the Exponential and Weibull. The parameter estimates obtained by classical approach. along with Akaike Information Criteria (AIC) ([24]) and Bayesian Information Criteria (BIC) ([25]) for Exponential distribution and Weibull distribution are given in the following Table 10.

Table 10 Parameter estimates with AIC and BIC values for Motor Data.

	q=2	
	Exponential	Weibull
Parameter Estimates	$\hat{\theta}$=0.0030, $\hat{\beta_1}$= 6.4960	$\hat{a}$=2.7973 , $\hat{b}$=235.4849, $\hat{\beta_1}$= 2.0637
AIC Values	361.9834	343.215
BIC Values	366.6545	345.8861
	q=4	
Parameter Estimates	$\hat{\theta}$=0.0033, $\hat{\beta_1}$= 5.3498	$\hat{a}$=2.8728, $\hat{b}$= 219.4164, $\hat{\beta_1}$= 1.5636
AIC Values	323.039	306.5712
BIC Values	327.7101	309.2424
	q=5	
Parameter Estimates	$\hat{\theta}$=0.0035, $\hat{\beta_1}$= 6.2985	$\hat{a}$=3.2639 , $\hat{b}$=204.5529, $\hat{\beta_1}$= 1.9613
AIC Values	274.7429	255.7912
BIC Values	279.414	258.4623

Table 10 reveals that a lower AIC or BIC value, indicative of an Weibull distribution, signifies a better fit. Therefore, we opt for the Weibull model based on the AIC and BIC values presented in the table above. And as censoring increases the model fits better. The associated reliability of the model corresponding to the above said censoring schemes are $0.5738, 0.8431$ and 0.9999 respectively.

6. Conclusions

In conclusion, this study provides valuable understandings regarding parameter estimation methods for parallel-connected subsystems with k-out-of-m load sharing arrangements. By utilizing Exponential and Weibull baseline distributions and accounting for censored Type-I and Type-II failure data, the effectiveness of these methods in precisely estimating load sharing parameters is demonstrated. Through thorough simulations and validation using actual data, the research confirms the reliability and practical usefulness of the proposed technique. Additionally, the adaptability of the approach is emphasized, indicating its applicability to a range of load-sharing configurations beyond those initially examined. The estimation can also be applicable for various other load sharing configurations, including series setups or following a p-out-of-q arrangement.

Acknowledgment

The Santosh S. Sutar is grateful for the financial support provided by Department of Science and Technology (DST), Science and Engineering Research Board (SERB)', Government of India under 'Core Research Grant (CRG)' (FILE NO.CRG/2021/005672-G). We also sincerely thank two anonymous referees for their valuable comments and suggestions on a previous version of this paper.

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